On Simple Singular N-Flat Modules

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ABSTRACT

Let I be a right ideal of a ring R , then R/I is right N-flat module if and only if for each $a \in I$, there exists $b \in I$ and a positive integer n such that $a^n \neq 0$ and $a^n = ba^n$ .In this paper, we first introduce and characterize rings whose every simple singular right R-module is N - flat. Next, we investigate the strong regularity of rings whose every simple singular right R - module is N-flat. It is proved that :

R is strongly regular ring if and only if R is a wjc , MERT and 2 - primal ring whose simple singular right R- module is N - flat.

Let R be a wjc ring satisfying condition (*). If every simple singular right R-module is N-flat .Then, the Center of R is a regular ring.

Keywords: N-Flat , MC2 - ring , WPSI ring , GQ - injective , CAM - ring

حول المقاسات البسيطة المنفردة المسطحة من النمط N

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الملخص

ليكن I مثالي أيمن في الحلقة R ، فإن R/I يكون مقاساً مسطحاً من النمط – N أيمن إذا وفقط إذا لكل $a \in I$, يوجد $b \in I$ وعدد صحيح موجب n بحيث إن $a^n \neq 0$ و $a^n = ba^n$. في هذا البحث, أولاً نقدم ونميز الحلقات التي فيها كل مقاس منفرد بسيط مسطحاً من النمط – N . ونقوم بالبحث عن الحلقات المنتظمة بقوة والتي فيها كل مقاس منفرد بسيط مسطحاً من النمط – N . أما أبرز النتائج التي حصلنا عليها :

الحلقة R تكون منتظمة بقوة إذا وفقط إذا كانت R من النمط wjc , MERT وحلقة أولية من النمط – 2 والتي فيها كل مقاس منفرد بسيط مسطحاً من النمط – N أيمن .

لتكن R حلقة من النمط wjc تحقق الشرط (*). إذا كان كل مقاس منفرد بسيط مسطحاً من النمط – N أيمن. فإن مركز الحلقة R يكون منتظماً.

الكلمات المفتاحية: مسطحة من النمط–N، حلقة من النمط–MC2، حلقة WPSI، الغامرة من النمط–GQ، حلقة من النمط–CAM.

1. Introduction

Throughout this paper , R denotes an associative ring with identity, and all modules are unitary right R - module. For any nonempty subset X of a ring R, $r(X)$ and $l(X)$ denote the right annihilator of X and the left annihilator of X, respectively . If $X = \{a\}$, we use the abbreviation $l(a)$ and $r(a)$. We write $J(R)$, $Z(R)$ and $Y(R)$ for the Jacobson radical of R , the left singular ideal of R and the right singular ideal of R, respectively.

Recall that , Let I be a right ideal of a ring R, then R/I is a flat right R - module if and only if for each $a \in I$, there exists $b \in I$ such that $a = ba$ (cf. [7] ,[8],[3]). The generalization of flat module to N-flat module is performed as follows : Let I be a right ideal of a ring R, then R/I is a right N-flat module if and only if for each $a \in I$, there exists $b \in I$ and a positive integer n such that $a^n \neq 0$ and $a^n = ba^n$ [1] . And , in [1] , we give a lot of characterizations of right N - flat . For example , $J(R) \cap Y(R) = (0)$, if R satisfying condition (*) whose every simple singular right R-module is N-flat .

The ring R is said to be 2 - primal if N(R) = P(R) , where N(R) is the set of all nilpotent element and P(R) is the prime radical of R [4] . The ring R is called right wjc ring , if $aRb = 0$ for $a,b \in R$, implies $bRa = 0$ [12] . The ring R is said to be reduced if R has no nonzero nilpotent element. The ring R is called right SXM [12] , if for each $0 \neq a \in R$, $r(a) = r(a^n)$ for all a positive integer n satisfying $a^n \neq 0$. For example, reduced rings are right SXM rings. The concept of regular rings was introduced in (1936) by Von Neumann [9] . The ring R is called MERT , if every essential maximal right ideal of R is an ideal . The ring R is called regular (strongly regular) ring , if for every $a \in R$, there exists $b \in R$ such that $a = aba$ ($a = a^2b$) .The ring R is called right weakly regular ring , if for every $a \in R$, $a \in aRaR$ [7] .

2. Characterizations of Simple Singular R - Module is N - Flat

This section is devoted to study rings whose every simple singular right R-module is N-flat with some of their basic properties . On the other hand ,we characterized MC2 rings terms of simple singular right R - modules is N - flat .

Definition 2.1 [10]

The ring R is called right MC2 ring . If $eRa = 0$ implies $aRe = 0$, where $a \in R$, $e^2 = e \in R$ and $K \cong eR$ is minimal right ideal of R , or equivalently if eR are minimal , $e^2 = e \in R$; then $K = gR$ for some $g^2 = g \in R$.

We consider the condition (*) ; R satisfies $l(a) \subseteq r(a)$ for every $a \in R$.

Theorem 2.2

Let R be a right MC2 ring satisfying condition (*). If every simple singular right R - module is N - flat , then $Z(R) = (0)$.

Proof : Suppose that $Z(R) \neq (0)$, then there exists $0 \neq a \in Z(R)$ such that $a^2 = 0$.We claim that $Z(R) + r(a) = R$. Otherwise , there exists a maximal right ideal M such that $Z(R) + r(a) \subseteq M$. If M is not essential , then $M = r(e)$, $e^2 = e \in R$. Hence $ea = 0$ because $a \in r(a) \subseteq M = r(e)$. If $eRa \neq 0$, then $eRaR = eR$ because eR is a minimal right ideal of R. Since $a \in Z(R)$, $eRaR \subseteq Z(R)$, then $e \in Z(R)$, which is a contradiction. Hence , $aRe = 0$.Since R right MC2 ring , then $aRe = 0$,

$e \in r(a) \subseteq M \subseteq r(e)$, which is a contradiction . Hence , M is essential in R . Thus , R/M is N - flat , then there exists a positive integer n and $b \in M$ such that $a^n \neq 0$ and $a^n = ba^n$. Since $a^2 = 0$, then $n = 1$ and therefore $(1-b) \in r(a) \subseteq M$ and $1 \in M$, which is a contradiction . Hence , $Z(R) + r(a) = R$. Write $1 = x + y$, where $x \in Z(R)$ and $y \in r(a)$, then $a = ax$. Since $x \in Z(R)$ and $l(x) \cap l(1-x) = 0$, $l(1-x) = 0$. Thus , $a = 0$ because $a \in l(1-x)$, which is a contradiction. This implies that $Z(R) = (0)$.■

Theorem 2.3

Let R be a ring whose every simple singular right R-module is N - flat, satisfying condition (*) .Then , $Z(R) \cap Y(R) = (0)$.

Proof : Suppose that $Z(R) \cap Y(R) \neq (0)$, then there exists $0 \neq a \in Z(R) \cap Y(R)$ such that $a^2 = 0$. We claim that $Z(R) + r(a) = R$. Otherwise, there exists a maximal essential right ideal M such that $Z(R) + r(a) \subseteq M$. Thus , R/M is N - flat ,then there exists $b \in M$ and a positive integer n such that $a^n \neq 0$ and $a^n = ba^n$. Since $a^2 = 0$, then $n = 1$, and therefore $a = ba$, which implies that $(1-b) \in l(a) \subseteq r(a) \subseteq M$, $1 \in M$, which is a contradiction . Hence , $Z(R) + r(a) = R$. Write $1 = x + y$,, $x \in Z(R)$ .Thus , $l(1-x) = 0$, $l(x) \cap l(1-x) = 0$ and $x \in Z(R)$. Since $a = ax$, then $y \in r(a)$ This implies that contradiction. a, which is $a \in l(1-x)$ because $a = 0$ ■ . $Z(R) \cap Y(R) = (0)$

Lemma 2.4 [1]

If R is a ring satisfying condition (*) whose every simple singular right R- module is N - flat , then $J(R) \cap Y(R) = (0)$.■

According to [11] ,a right R - module M is said to be right weakly principally small injective (WPSI) if for any $0 \neq a \in J(R)$, there exists a positive integer n such that $a^n \neq 0$ and any right R - homomorphism from $a^n R \to M$ can be extended to $R \to M$. A ring R is called right WPSI if R_R is a right WPSI.

Definition 2.5 [6]

The ring R is said to be right(left) mininjective if every R - homomorphism from a minimal right(left) ideal of R to R_R can be extended from R to R_R .

Lemma 2.6 [11]

If R is a left (right) WPSI ring , then :

1. $J(R) \subseteq Z(R)$ ($J(R) \subseteq Y(R)$) .
2. R is a left (right) mininjective ring .■

Corollary 2.7

Let R be a right MC2 ring , left WPSI satisfying condition (*). If every simple singular R - module is N - flat , then $J(R) = (0)$.

Proof : From Lemma 2.6 , we have $J(R) \subseteq Z(R)$. So $J(R) = (0)$ because $Z(R) = (0)$, by Theorem 2.2 .■

Proposition 2.8

Let R be a right WPSI ring , whose every simple singular right R- module is N - flat and satisfying condition (*) .Then , $Z(R) = (0)$.

Proof : By Lemma 2.4 , $J(R) \cap Y(R) = (0)$. Since R is a right WPSI ring then , $J(R) \subseteq Y(R)$ Lemma 2.6 , so $J(R) = J(R) \cap Y(R) = (0)$ and R is a right mininjective ring Lemma 2.6.Therefore,R is a right MC2 ring [5,Theorem 1.14]. Hence , by Theorem 2.2 . $Z(R) = (0)$

3. Certain Rings Whose every Simple Singular R - Module is N - Flat

In this section , we study the relation between rings whose every simple singular right R - modules are N - flat , reduced rings and strongly rings by adding some types of rings such as wjc and MC2 rings , and other types of rings .

Theorem 3.1
Let R be a wjc ring satisfying condition (*) .If every simple singular right R- module is N - flat . Then , R is a semiprime .

Proof : Suppose that I is an ideal of R with $I^2 = 0$. If $I \neq 0$, then there exists $0 \neq a \in I$ such that $aI = 0$ and so $a^2 = 0$. First observe r(a) which is an essential right ideal of R. if not , then there exists a nonzero right ideal K of R such that $r(a) \cap K = 0$.Since $aKI \subseteq aI = 0$, $KI \subseteq r(a) \cap K = 0$.Since R is a wjc ring , $IK = 0$. Hence , $aK = 0$ and so $K \subseteq r(a) \cap K = 0$, which is a contradiction .Hence, r(a) is essential and $r(a) \neq R$, Thus , there exists a maximal essential right ideal M of R containing r(a). Hence , R/M is simple singular R - module and so is N - flat . Since R/M is N- flat , then there exists $d \in M$ and a positive integer n such that $a^n \neq 0$ and $a^n = da^n$, since $a^2 = 0$, then $n = 1$, so that $a = da$ and we get $(1-d) \in l(a) \subseteq r(a) \subseteq M$, where $1 \in M$, which is also a contradiction . Hence , $I = 0$ and so R is a semiprime .■

Definition 3.2 [13]
The ring R is called strongly right min-abel if every right minimal idempotent element $e \in R$, is a left semicentral element .
We now consider other condition for right simple singular R-module N-flat to be semiprime.

Theorem 3.3
Let R be a strongly right min - abel ring satisfying condition(*). If every simple singular right R - module is N - flat , then R is a semiprime ring.

Proof : Let $a \in R$ satisfyies $aRa = 0$. Suppose that $a \neq 0$. Then , there exists a maximal right ideal M is of R containing r(a). If M is not essential in R .Then , M is a direct summand of R , since M is maximal . So we can write $M = r(e)$ for some $0 \neq e^2 = e \in R$, $b \in R$ and hence $eb = 0$. Because eR is a minimal right ideal of R and R is a strongly right min-abel ring, $be = ebe = 0$ .Thus , $e \in r(b) \subseteq M = r(e)$, whence $e = 0$. This is a contradiction. Therefore , M must be an essential right ideal of R .Thus , R/M is N-flat and so there exists $d \in M$ such that $a = da$ implies that $(1-d) \in l(a) \subseteq r(a) \subseteq M$ whence $1 \in M$ .This is also a contradiction . Hence , R is a semiprime ring .■

Theorem 3.4

Let R be a strongly right min - able with $l(a^n) \subseteq r(a)$ for every $a \in R$ and appositive integer n . If every simple singular right R - module is N - flat . Then , R is a right weakly regular ring .

Proof : We show that $RaR + r(a) = R$ for any $a \in R$. Suppose that there exists $b \in R$ such that $RbR + r(b) \neq R$. Then , there exists a maximal right ideal M of R containing $RbR + r(b)$. If M is not essential in R . Then , M is a direct summand of R because M is maximal .

By a similar method of proof used in Theorem 3.3 , M is an essential right ideal of R . Thus , R/M is N - flat and so there exists a positive integer n such that $b^n \neq 0$ and $b^n = cb^n$ for some $c \in M$, implies that $(1-c) \in l(b^n) \subseteq r(b) \subseteq M$, whence $1 \in M$, a contradiction . Therefore , R is a right weakly regular . ■

According to [1], rings satisfying condition (*), and right GQ -injective whose every simple singular right R- module is N-flat are always regular .But ,in general rings satisfying condition (*) and left GQ - injective whose every simple singular right R-module is N-flat need not be regular .This leads to the following theorem:

Theorem 3.5

Let R be a left GQ - injective , right MC2 ring whose every simple singular right R - module is N - flat and satisfying condition (*).Then , R is a regular ring .

Proof : First R is left GQ - injective , then $Z(R) = J(R)$ and $R/J(R)$ is a regular ring . From Theorem 2.2 , $Z(R) = (0)$. Hence , $J(R) = Z(R) = (0)$, which implies R is a regular ring .■

Definition 3.6 [2]

R is called a right CAM-ring , if for any maximal essential right ideal M of R (if it exits) and for any right subideal I of M which is either a complement right subideal of M or a right annihilator ideal in R , I is an ideal of M.

Right CAM-ring generalizes semisimple artinian . [2] shows that semiprime right CAM - ring R is either semisimple artinian or reduced . If R is also simple singular right R - module is N - flat , then R is either semisimple artinian or strongly regular ring. We yield the following theorem :

Theorem 3.7

Let R be a wjc right CAM-ring , satisfying condition (*) whose every simple singular right R - module is N - flat .Then , R is either a semisimple artinian or strongly regular ring.

Proof : From Theorem 3.1 R is semiprime .If R is not a semisimple artinian , then R is reduced .Let $0 \neq a \in R$. If $aR \oplus r(a) \neq R$, then $aR \oplus r(a) \subseteq M$ for some maximal right ideal M of R . If M is not an essential right ideal of R , then $M = eR$, where $e^2 = e \in R$. Because R is reduced , $ae = ea = 0$ and $e \in r(a) \subseteq M = r(e)$, a contradiction . Hence , M is an essential right ideal of R and so R/M is a singular simple right R - module . Since R/M is N - flat , then there exists $b \in M$ and a positive integer n such that $a^n \neq 0$ and $a^n = ba^n$. Now , we obtain $(1-b) \in r(a)$, so $1 \in M$, a contradiction . Hence , $aR \oplus r(a) = R$ and then R is a strongly regular ring .■

An idempotent $e \in R$ is called right semicentral if $ea = eae$ for all $a \in R$ [13].

Lemma 3.8 [12]

Let R be a MERT ring and every right minimal idempotent in $R/J(R)$ is right semicentral .Then , $R/J(R)$ is reduced .■

Proposition 3.9

Let R be a MERT ring and every right minimal idempotent of $R/J(R)$ is right semicentral . If every simple singular right R - module is N - flat. Then , $R/J(R)$ is a strongly regular ring.

Proof : By Lemma 3.8 , $R/J(R)$ is reduced . Let $\bar{0} \neq \bar{a} \in \bar{R} = R/J(R)$. We first that $\bar{R}\,\bar{a}\,\bar{R} + r(\bar{a}) = \bar{R}$. Suppose that it is not ; then there exists a maximal right ideal M of R such that $\bar{R}\,\bar{a}\,\bar{R} + r(\bar{a}) \subseteq M/J(R)$. Since R' is reduced , $\bar{R}\,\bar{a}\,\bar{R} + r(\bar{a})$ is an essential right ideal of R' . Therefore , R/M is simple singular right R- module and so R/M is N -flat .
Now, it is easy to show that there exists a positive integer n and $c \in M$ such that $a^n \neq 0$ and $a^n = ca^n$. Since M is an ideal of R , we have $1 \in M$, which is a contradiction . Hence , $\bar{R}\,\bar{a}\,\bar{R} + r(\bar{a}) = \bar{R}$. Let $\bar{a} = \bar{a}\,\bar{d}$, where $\bar{d} \in \bar{R}\,\bar{a}\,\bar{R}$. Then , $(\bar{1} - \bar{d}) \in r(\bar{a})$. If $\bar{a}\,\bar{R} + r(\bar{a}) \neq \bar{R}$, then there exists a maximal right ideal L of R such that $\bar{a}\,\bar{R} + r(\bar{a}) \subseteq L/J(R)$. Since $R/J(R)$ is reduced , we have $L/J(R)$ is an ideal of $R/J(R)$. Hence , L is also an ideal of R, and so $\bar{d} \in \bar{R}\,\bar{a}\,\bar{R} \subseteq L/J(R)$, and hence $\bar{1} \in L/J(R)$, which is also a contradiction . Thus , $\bar{a}\,\bar{R} + r(\bar{a}) = \bar{R}$, and so $R/J(R)$ is strongly regular ring.■

Lemma 3.10 [4]

For any $a \in Cent(R)$ (The Center of R) , if $a = ara$ for some $r \in R$; Then there exists $b \in Cent(R)$ such that $a = aba$.■

Lemma 3.11 [4]

If R is a semiprime ring ;then $r(a^n) = r(a)$ for any $a \in Cent(R)$ and a positive integer n .■

Proposition 3.12

Let R be a wjc ring satisfying condition (*) . If every simple singular right R - module is N - flat . Then , the Center of R is a regular ring.

Proof : Let a be a nonzero element in Cent(R) . First , we will show that $aR + r(a) = R$ for any $a \in Cent(R)$. If not, there exists a maximal right ideal M of R such that $aR + r(a) \subseteq M$. Since $a \in Cent(R)$, $aR + r(a)$ is an essential right ideal and so M must be an essential right ideal of R. Therefore , R/M is N - flat , so there exists $c \in M$ and a positive integer n such that $a^n \neq 0$ and $a^n = ca^n = a^n c$ implies that $(1-c) \in r(a^n) \subseteq r(a) \subseteq M$ (Theorem 3.1 and Lemma 3.11) and so $1 \in M$, which is a contradiction . Therefore , $aR + r(a) = R$ for any $a \in Cent(R)$ and so we have $a = ara$ for some $r \in R$. Applying Lemma 3.10 , Cent(R) is a regular ring .■

Theorem 3.13

R is strongly regular ring if and only if R is a wjc , MERT and 2 - primal ring whose every simple singular right R- module is N - flat.

Proof : First , we show that R is reduced . In fact , if $a^2 = 0$ for some $0 \neq a \in R$. Then , we have $RaR + r(a) = R$. If not , then there exists a maximal right ideal M of R containing $RaR + r(a)$. Observe that M must be an essential right ideal of R. If not , then M is a direct summand of R . So we can write $M = r(e)$ for some idempotent e of R. Thus $eRa = 0$. Since R is a wjc ring , $aRe = 0$ and $ae = 0$. Hence , $e \in r(a) \subseteq r(e)$; whence $e = 0$, it is a contradiction . Therefore , M must be an essential right ideal of R. Then , R/M is N - flat ,there exists a positive integer n and $b \in M$ such that $a^n \neq 0$ and $a^n = ba^n$. Since $a^2 = 0$, then $n = 1$ and therefore $a = ba$ which implies $(1 - b) \in M$ and so $1 \in M$ by M is an ideal of R . This is a contradiction . Hence $RaR + r(a) = R$ and so $a = ad$ for some $d \in RaR$. Since R is 2 - primal ring , $d \in J(R)$. Hence , $(1 - d)$ is right inveritable v in R such that $v(1 - d) = 1$, $v(a - da) = a$ which yield $a = 0$.

Next, we show that $aR + r(a) = R$ for each $a \in R$. If not , then there exists $b \in R$ and a maximal right ideal L of R containing $bR + r(b)$. Observe that L must be essential , so there exists a positive integer n and $d \in M$ such that $a^n \neq 0$ and $a^n = da^n$. Now , $(1 - d) \in l(a^n) = r(a^n) = r(a) \subseteq L$. so $1 \in L$, which is a contradiction , therefore $aR + r(a) = R$. Hence , R is strongly regular ring .

Conversely : it is obvious. ■

REFERENCES

[1] Mahmood, R.D. and Husam Q.M. (2010) ; "On N-flat rings", Received, J. Al - Rafiden of Comp. and Math.

[2] Ming , R.Y.C. (1983) ; " On Quasi-Frobeniusean and artinian Rings", Pub. De L'institute Mathématique, 33(47) , pp. 239-245.

[3] Ming , R.Y.C. (1995); " On Von Neumann, regularity, injectivity and flatness" , Yokohama Math. , Vol. 43, pp.37-44 .

[4] Nam, S.B. (1999) ; "A note on simple singular GP-injective modules", J. Kangweon - Kyungki. Mth. J. 7(2), pp.215-218.

[5] Nicholson , W.K. and Yousif , M. F. (1997) ;" Mininjevtive rings " , J. Algebra , Vol. 187 , pp. 548-578.

[6] Nicholson , W.K. and Yousif M.F. (2004) ; " Weakly continuous and C2 - rings" , Comm. In Algebra , pp. 2429-2466.

[7] Ramamurthi, V.S. (1975); "On injectivity and flatness of certain cyclic modules" , Proc. Amer. Math. Soc., 48, pp.21- 25.

[8] Rege , M.B. (1986) ; " On Von Neumann regular rings and SF- rings" , Math. Japonica , 31(6), pp. 927-936.

[9] Von Neumann J. (1936) ; " On regular rings " , Proc. Nat. Sci. U.S.A. 22, pp. 707-713.

[10] Wei, J.C. (2005) ; " The rings characterized by minimal left ideal" , Acta. Math. Sinica , English Series 21(3) , pp. 473- 482.

[11] Wei , J.C. and Chen , J.H. (2007) ; " Nil-injective rings " , Inter . J. of Algebra 2, pp. 1-21.

[12] Wei , J.C. (2007); "On simple singular YJ- injective modules", Sou. Asiam Bull. Of Math. 31, pp. 1 - 10.

[13] Wei , J.C. (2008); "Certain rings whose simple singular modules are nil-iniective" , Turk J. Math .32, pp. 393 - 408.