A kind of Upwind Finite Element Approximations for Compressible Flow of Contamination from Nuclear Waste in Porous Media

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ABSTRACT

A non-linear parabolic system is derived to describe compressible nuclear waste disposal contamination in porous media . Galerkin method is applied for the pressure equation . For the concentration of the brine of the fluid, a kind of partial upwind finite element scheme is constructed. A numerical application is included to demonstrate certain aspects of the theory and illustrate the capabilities of the kind of partial upwind finite element approach.

Keywords: Galerkin method; A kind of partial upwind finite element scheme; Non-linear parabolic system

تقريبات نوعاً من العناصر المنتهية الجزئية لحل نموذج ناشئ من مخلفات الطاقة لحالة قابلة للانضغاط خلال مناطق مسامية

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الملخص

في منظومة المعادلات اللاخطية التي تصف مخلفات الطاقة في المناطق المسامية لحالة قابلة للانضغاط، تم استخدام طريقة كلاركن لحل معادلة الضغط وكذلك تم استخدام تقنية جديدة تستعمل نوعاً من العناصر المنتهية الجزئية لحل معادلة التركيز (brine) وذلك بتخمين الخطأ الناتج من القيمة التقريبية لها. وتضمن الجانب العملي حقائق تأكد ما عملناه في الجانب النظري.

1. Introduction

The proposed disposal of high-level nuclear waste in underground repositories is an important environmental topic for many countries. Decisions on the feasibility and safety of the various sites and disposal methods will be based, in part, on numerical models for describing the flow of contaminated brines and groundwater through porous or fractured media under severe thermal regimes caused by the radioactive contaminants.

A fully discrete formulation is given in some detail to present key ideas that are essential in code development . The non-linear couplings between the unknowns are important in modeling the correct physics of flow .

In this model one obtain a convection-diffusion equations which represent a mathematical model for a case of diffusion phenomena in which underlying flow is present ; Δw and $b\nabla w$ correspond to the transport of w through the diffusion process and the convection effects, respectively, where ∇ and Δ denoted respectively the gradient operator and the Laplacian operator in the spatial coordinates.

In this paper we will consider the fluid flow in porous media using a Galerkin method for the pressure equation and a kind of partial upwind finite element scheme is constructed for the convection dominated saturation (or concentration) equation. For more details of this subject see [7, 6, and 4].

2. Model Equations

The model for compressible flow and transport of contaminated brine in porous media can be described by a differential system that can be put into the following form [2].

Fluid:

$$\phi_1 \frac{\partial p}{\partial t} + \nabla.u = -q + R_s'$$

$$u = -\frac{k(x)}{\mu(c)}\nabla P = -a(c)\nabla P$$

$$...(1)$$

Brine:

$$\phi \frac{\partial c}{\partial t} + u.\nabla c - \nabla.(E_c \nabla c) = g(c) \qquad ...(2)$$

Radionuclide :

$$\phi K_i \frac{\partial c_i}{\partial t} + u.\nabla c_i - \nabla.(E_c \nabla c_i) + d_3(c_i)\frac{\partial p}{\partial t} = f_i(c, c_1,....,c_N) \qquad ...(3)$$

Heat:

$$d_1(p)\frac{\partial p}{\partial t} + d_2\frac{\partial T}{\partial t} + c_p u.\nabla T - \nabla.(\tilde{E}_H \nabla T) = Q(u, T, c, p) \qquad ...(4)$$

Where $x \in \Omega \subset R^2, t \in (0, \bar{T}]$, p=p(x,t) is the pressure, T =T(x,t) is the temperature of the fluid, c =c(x,t) is the concentration of the brine in the

fluid , $c_i = c_i(x,t)$ is the trace concentration of the i-th radionuclide, i=1,2,…,N, u is the Darcy velocity, $\phi_1 = \phi c_w, \phi = \phi(x), c_w$ is the concentration of the rock, $\mu(c)$ is the viscosity of the fluid, $q = q(x,t)$ is a production term, $R_s' = R_s'(c) = [c_s \phi K_s f_s /(1+c_s)](1-c)$ is a salt dissolution term, see [3].

$$d_2 = \phi c_p + (1-\phi)\rho_R c_{pR}, d_1(p) = -\phi c_w[U_0 + (p/\rho)], d_3(c_i) = \phi c_w c_i(K_i - 1)$$

$$\tilde{E}_H = D c_{pw} + K_m' I, K_m' = k_m/\rho_0, D = (D_{ij}) = (\alpha_T |u| \delta_{ij} + (\alpha_L - \alpha_T) u_i u_j /|u|),$$

and $Q(u,T,c,p) = -\{[\nabla U_0 - c_p \nabla T_0].u + [U_0 + c_p(T - T_0)) + (p/\rho)][-q + R_s']\}$

$$- q_L - qH - q_H.$$

$E_c = D + D_m I,$ and g(c) = -c$\{[c_s \phi K_s f_s /(1+c_s)](1-c)\} - q_c + R_s'$,

and

$$f_i(c,c_1,c_2,...,c_N) = c_i\{q - [c_s \phi K_s f_s /(1+c_s)](1-c)\} - qc_i - q_{ci} + q_{oi}$$

$$+ \sum_{j=1}^{N} k_{ij} \lambda_j K_j \phi c_j - \lambda_i K_i \phi c_i.$$

Shifting with the boundary conditions

(a) $u.n = 0$ on Γ

(b) $(E_c \nabla c - cu).n = 0$ on Γ

(c) $(E_c \nabla c_i - c_i u).n = 0$ on Γ

(d) $(E_H \nabla T - c_p Tu).n = 0$ on Γ

(e) $\dfrac{\partial p}{\partial n} = o$; $(x,t) \in \Gamma \times (0,\bar{T}]$

...(5)

Shifting and the initial conditions

(a) $p(x,0) = p_0(x)$; $x \in \Omega$

(b) $c(x,0) = c_0(x)$; $x \in \Omega$

(c) $c_i(x,0) = c_{0i}(x)$; $x \in \Omega$

(d) $T(x,0) = T_0$; $x \in \Omega$

...(6)

     The reservoir Ω will be taken to be of unit thickness and will be identified with a bounded domain in R^2 . We shall omit gravitational terms for simplicity of exposition, no significant mathematical questions arise the lower order terms are included.

We assume that

(A1) $a(c), R_s'(c), g(c), E_c \in C_0^1(R)$,

        $\phi(x), \phi_1(x), E_c \in H'(\Omega), q \in L^\infty(0,T;H^1(\Omega))$

$$0 < c_0 \leq a(c), R_s^{'}(c), g(c), \phi, \phi_1, E_c \leq c_1 \ , \forall c_0, c_1 \in R, x \in \Omega$$

$$D_m > 0 \ and \ \alpha_l \geq \alpha_t > 0$$

(A2) The solution of the problem (from eq.(1) to eq.(6)) is regular:

$$c(x,t) \in L^2(0,T;H^2(\Omega)) \bigcap L^\infty(0,T;w_\infty^1(\Omega))$$

$$p(x,t) \in L^\infty(0,T;H^{r+1}(\Omega)) \ , \ (r \geq 2)$$

$$c_t, c_{tt}, c_{ttt} \in L^\infty(0,T;H^1(\Omega)) \ ; p_t, p_{tt} \in L^\infty(0,T;L^\infty(\Omega))$$

(A3) For any $\phi \in L^2(\Omega)$, the boundary value problem:

$$-\Delta\varphi + \varphi = \phi \ , \ x \in \Omega$$

$$\frac{\partial\varphi}{\partial n} = 0 \ , \ x \in \Gamma$$

there exists a unique solution $\varphi \in H^2(\Omega)$ and a positive constant M such that $\|\varphi\|_2 \leq M\|\phi\|$ .see [8]

3. Finite Element Spaces

Consider a regular family $\{T_h\}$ of triangulation defined over Ω, where h is the longest diameter of a triangular element with the triangular T_h, we have a set of close triangles $\{e_i\}(1 \leq i \leq N_e)$ and a set of nodes $\{P_i\}(1 \leq i \leq N_P + M_P)$ where $P_i(1 \leq i \leq N_P)$ are interior nodes in Ω and $P_j(N_{P+1} \leq j \leq N_P + M_p)$ are boundary nodes on Γ. We put h_s to be the maximum side length of triangles and k to be minimum perpendicular length of triangles for all $e \in T_h$.

Definition(3.1): A family T_h of triangulation is of weakly acute type, if there exists a constant $\theta_0 > 0$ independent of h such that, the internal angle θ of any triangle $e_i \in T_h$ satisfies $\theta_0 \leq \theta \leq \pi/2$.

Let $\phi_i(p),(1 \leq i \leq M)$, be the continuous function in Ω s.t. $\phi_i(p)$, is linear on each $e \in T_h$ and $\phi_i(p_j) = \delta_{ij}$ for any nodal point p_j..We denote M_h the linear span of $\phi_i,(1 \leq i \leq M)$,i.e.,a finite dimensional subspace of $H^1(\Omega)$

$M_h = \{z_h \,|\, z_h \in C(\Omega); z_h \ is a linear function on e, \forall e \in T_h\}$.

And a subspace of $H_0^1(\Omega)$

$M_{0h} = \{z_h \,|\, z_h \in M_h; z_h(P_k) = 0, k = M + 1,......,K\}$.

We associate the index set $\Lambda = \{j \neq i : P_j \ is \ adjacent \ to \ P_i\}$ .Let P_i P_j ,P_k , be three vertices of triangular element e and $\lambda_i, \lambda_j, \lambda_k$ be barycentric coordinates. We have the following definitions see [5].

Definition(3.2): with each vertex P_i belonging to triangle e, the barycentric subdivision Ω_i^e is given by:

$\Omega_i^e = \{P \mid P \in e \; ; \; \lambda_i(P) \geq \lambda_j(P) \; , \; \lambda_i(P) \geq \lambda_k(P), \forall P_j \in e\}$, and the barycentric domain Ω_i associated with vertex P_i in Ω is given by $\Omega_i = \cup \Omega_i^e \; , e \in T_h$.

Definition(3.3): with the characteristic function $\mu_i(x)$ of barycentric domain Ω_i, the mass lumping operator $\wedge : w \in C(\Omega) \to \hat{w} \in L_\infty(\Omega)$ is defined by $\hat{w}(p) = \sum_i^{N_p + M_p} w(p_i) \mu_i(p)$.

Definition(3.4): Let $\{M_h\}$ be a family of finite dimensional subspaces of $C(\Omega)$, which is piecewise polynomial space of degree less or equal to r with step length h_P and the following property: for $P \in [1, \infty)$, $r \geq 2$, there exists a constant M such that for $0 \leq q \leq 2$ and $\phi \in w_P^{r+1}(\Omega)$:

$\inf_{x \in \{M_h\}} \|\phi - x\|_{q,p} \leq M h^{r+1-q} |\phi|_{r+1,p}$

Similarly, we define $\{N_h\}$ be a family of finite-dimensional subspace of $C(\Omega) \times C(\Omega)$, which is piecewise polynomial space of degree less or equal to r-1 with the similar property as M_h and $0 \leq q \leq r - 1$. We also assume the families $\{M_h\}$ and $\{N_h\}$ satisfy inverse inequalities:

$$\|\phi\|_{L^\infty} \leq M h_P^{-1} \|\phi\| \qquad \|\nabla \phi\|_{L^\infty} \leq M h_P^{-1} \|\nabla \phi\| \qquad \forall \phi \in M_h$$

see [8].

Lemma(3.1): [5]

There exists a constant C such that:

$\|w - \hat{w}\|_{o,p} \leq C h_c |w|_{1,p}$, $\forall w \in M_h$, $p \geq 1$ ...(7)

$|w_h|_1 \leq M h_c^{-1} \|w_h\|$, $\forall \; w_h \in M_{0h}$ ...(8)

Lemma(3.2): [8]

There exists constants $C_1, C_2 > 0$ such that:

$C_1 \|w\|_{0,p} \leq \|\hat{w}\|_{0,p} \leq C_2 \|w\|_{0,p}$, $\forall w \in M_h$ ...(9)

Lemma(3.3): [10]

Let $\bar{p} \in V_h$ be the elliptic projection of $p \in H^1(\Omega)$ into V_h defined by $(a(c)\nabla\bar{p}, \nabla v) = (a(c)\nabla p, \nabla v)$, $\forall v \in V_h$ then there exists a constant k_1 such that

$\|p - \bar{p}\| + h_p \|\nabla p - \nabla \bar{p}\| \leq k_1 \|p\|_{r+1} h_p^{r+1}$

4. Error Estimates

Let $\tau > 0$ is time step and $N_\tau = T/\tau$. We use a Galerkin finite element method for the pressure and velocity and partial upwind finite element scheme for brine.

Let $C^0 \in M_h$ be a $L^2(\Omega)$ - projection of c^0 in M_h:

$$(c^0 - C^0, z_h) = 0 \quad \forall z_h \in M_h.$$

We can get $P^0 \in V_h$ such that

$$\int_\Omega P^0 dx = 0, \ (\frac{k(x)}{\mu(c^0)}\nabla P^0, \nabla v) = (-q^0, v) + (R'_s, v) \ , \ \forall v \in V_h \ and \ U^0 \in W_h \ from$$

$$U^0 = -\frac{k(x)}{\mu(C^0)}\nabla P^0 = -a(C^0)\nabla P^0$$

If the approximate solution $\{P^m, U^m, C^m\} \in V_h \times W_h \times M_h$ is known, we want to find $\{P^{m+1}, U^{m+1}, C^{m+1}\} \in V_h \times W_h \times M_h$ at $t = t^{m+1}$, with three steps. Let $(.,.)$ denote the inner product in $L^2(\Omega)$

Step 1. Find C^{m+1} for $m = 0,1,\ldots,N_\tau - 1$, such that

$$(\hat{\phi}D_\tau\hat{C}^m, \hat{z}_h) + (E_c\nabla C^{m+1/2}, \nabla z) + R(U^m, C^{m+1/2}, z_h) = (\hat{g}(C^{m+1/2}), \hat{z}_h) \ \forall z_h \in M_h \ \ldots(10)$$

where $D_\tau C^m = (C^{m+1} - C^m)/\tau$, $C^{m+1/2} = (C^{m+1} + C^m)/2$ and

$$R(U^m, C^{m+1/2}, z_h) = \sum_{i=1}^m z_i \sum_{j \in \Lambda i} \beta_{ij}^m (\alpha_{ij}^m C_i^{m+1/2} + \alpha_{ji}^m C_j^{m+1/2})$$

with $z_i = z_h(P_i)$, $C_i^{m+1/2} = C^{m+1/2}(P_i)$, and $\beta_{ij}^m = \int_{\Gamma_{ij}} U^m.n_{ij} \ d\Gamma$, here n_{ij} is the unit outer normal to Γ_{ij}. The partial upwind coefficients should be required that [5].

(a) $\alpha_{ij}^m + \alpha_{ji}^m = 1$

(b) $\max\{1/2, 1 - \rho_{ij}^{-1}\} \le \alpha_{ij} \le 1, if\beta_{ij} \ge 0,$ ...(11)

$\max\{1/2, 1 - \rho_{ij}^{-1}\} \le \alpha_{ji} \le 1, if\beta_{ij} < 0,$

step2. Find P^{m+1} such that:

$$\int_\Omega P^{m+1} dx = 0, \ (\frac{k(x)}{\mu(C^{m+1})}\nabla P^{m+1}, \nabla v) = (-q^{m+1}, v) + (R'_s, v) \ , \ \forall v \in V_h \qquad \ldots(12)$$

step3. Find U^{m+1} as :

$$U^{m+1} = -a(C^{m+1})\nabla P^{m+1} \qquad \ldots(13)$$

Let $\bar{c} : J \to M_h$ be determined by the relations

$$(E_c\nabla(c - \bar{c}), \nabla z) + \lambda(c - \bar{c}, z) = 0 \ , \ \forall z \in M_h \qquad \ldots(14)$$

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for $t \in J$, where the constant λ is chosen to be large enough to insure the coercivity of the bilinear form over $H^1(\Omega)$ Similarly, let $\bar{p} : J \to V_h$ satisfy

$$a(c)\nabla(p - \bar{p}), \nabla v) + \mu(p - \bar{p}, v) = 0 , \quad \forall v \in V_h \qquad \ldots(15)$$

where μ assures coercivity over $H^1(\Omega), and\ (\bar{p}, 1) = (p, 1)$. Let

$$\varsigma = c - \bar{c}, \ \xi = \bar{c} - C \quad , \quad \eta = p - \bar{p} , \ \pi = \bar{p} - P \qquad \ldots(16)$$

A standard result in the theory of Galerkin methods give [2].

$$\|\varsigma\| + h_c \|\varsigma\|_1 \leq M \|c\|_2 \, h_c^2$$

$$\|\eta\| + h_p \|\eta\|_1 \leq M \|p\|_{r+1} \, h_p^{r+1}$$

$$\|\bar{p}\|_{1,\infty} \leq M$$

for $t \in J$, where the constant M depends on bounds for lower order derivatives of p, c. And also

$$\left\|\frac{\partial \varsigma}{\partial t}\right\| + h_c \left\|\frac{\partial \varsigma}{\partial t}\right\|_1 \leq M\{\|c\|_2 + \left\|\frac{\partial c}{\partial t}\right\|_2\}h_c^2 \qquad \ldots(18a)$$

$$\left\|\frac{\partial \eta}{\partial t}\right\| + h_p \left\|\frac{\partial \eta}{\partial t}\right\|_1 \leq M\{\|p\|_{r+1} + \left\|\frac{\partial p}{\partial t}\right\|_{r+1}\}h_p^{r+1} \qquad \ldots(18b)$$

where M now depends on bounds for lower order derivatives of p, c and their first derivatives with respect to time .

Lemma (4.1): There exists a positive constant k_2 such that:
$$\left\|\nabla P^{m+1} - \nabla \bar{P}^{m+1}\right\| \leq k_2 \left\|C^{m+1} - c^{m+1}\right\|$$

Proof: see [1].

Lemma (4.2): For all $z_h \in M_h$ and $\varsigma = c - \bar{c}$, $\xi = \bar{c} - C$,

$$\left|((u^m . \nabla c^{m+1/2}), z_h) - R(U^m, C^{m+1/2}, z_h)\right| \leq M (h_c^2 + \left\|\xi^{m+1/2}\right\|^2 + \left\|\varsigma^{m+1/2}\right\|^2 +$$

$$\left\|\nabla \varsigma^{m+1/2}\right\|^2 + \left\|u^m - U^m\right\|^2 + \left\|z_h\right\|^2) + \varepsilon \left\|\nabla z_h\right\|^2$$

where $\varepsilon > 0$ is arbitrary small constant
Proof: [1].

Notes:

1.The inductive assumption [10],if $\left\|U^l\right\|_{L_\infty} \leq k^* \ (0 \leq l \leq m)$, then $\left\|U^{m+1}\right\|_{L_\infty} \leq k^*$

2.If T_h is regular triangulation of weakly acute type we have

$$|w_h|_1 \le \sqrt{6}/k \|\hat{w}_h\| \quad , \quad \forall w_h \in M_h \qquad , \text{ see [8]}.$$

Lemma (4.3): There exists a positive constant k_3 such that
$$\|U^{m+1} - u^{m+1}\| \le k_3 (\|C^{m+1} - c^{m+1}\| + h_P^r).$$

Proof:
$$\|U^{m+1} - u^{m+1}\| = \|a(C^{m+1})\nabla P^{m+1} - a(c^{m+1})\nabla P^{m+1}\|$$
$$\le \|a(C^{m+1})\nabla(P^{m+1} - p^{m+1})\| + \|a(C^{m+1}) - a(c^{m+1})\| \|\nabla p^{m+1}\|$$

From (A1) and (A2) we have
$$\le \text{const.}\|\nabla P^{m+1} - \nabla p^{m+1}\| + \|C^{m+1} - c^{m+1}\| \|\nabla p^{m+1}\|_{L_\infty}$$

we have $\|\nabla P^{m+1} - \nabla p^{m+1}\| \le \|\nabla P^{m+1} - \nabla \overline{p}^{m+1}\| + \|\nabla \overline{p}^{m+1} - \nabla p^{m+1}\|$

using lemma (3.3) and lemma (4.1) , we get
$$\|\nabla P^{m+1} - \nabla p^{m+1}\| \le k_2 \|C^{m+1} - c^{m+1}\| + k_1 \|p^{m+1}\|_{r+1} h_p^r, \text{ from (A2)},$$
$$\|U^{m+1} - u^{m+1}\| \le k_3 (\|C^{m+1} - c^{m+1}\| + h_P^r).$$

Theorem(4.1): For all $m \le l \le N_\tau$ if $\tau \le \tau_0$, then
$$\|c^{l+1} - C^{l+1}\| \le M(\tau + h_c + h_P^r), \text{ where M is independent of } \tau \text{ and } h_c.$$

Proof:
Multiply eq.(2) by z_h and integrating by parts we obtain for $t = (m+1/2)\tau$. Let $w_{m+1/2} = w(.,(m+1/2)\tau)$ and $w^{m+1/2} = (w^{m+1} + w^m)/2, then$

$$(\phi D_\tau c^m, z_h) + (Ec\nabla c^{m+1/2}, \nabla z_h) + (u^m \nabla c^{m+1/2}, z_h) =$$
$$(g(c^{m+1/2}), z_h) + (\phi(D_\tau c^m - \frac{\partial c}{\partial t}\big|_{m+1/2}), z_h) + \qquad\qquad ...(19)$$
$$+ ((Ec\nabla c^{m+1/2} - Ec\nabla c_{m+1/2}), \nabla z_h) + ((u^m \nabla c^{m+1/2} - u_{m+1/2}\nabla c_{m+1/2}), z_h)$$

Let $e = c - C = (c - \overline{c}) + (\overline{c} - C) = \varsigma + \xi$, and subtract (10) from (19), we obtain:
$$(\hat{\phi}D_\tau \hat{e}^m, \hat{z}_h) + (Ec\nabla e^{m+1/2}, \nabla z_h) = (R(U^m, C^{m+1/2}, z_h)$$
$$- (u^m \nabla c^{m+1/2}, z_h)) + ((\hat{\phi}D_\tau \hat{c}^m, \hat{z}_h) - (\phi D_\tau c^m, z_h))$$
$$+ ((g(c^{m+1/2}), z_h) - (\hat{g}(C^{m+1/2}), z_h)) + (\phi(D_\tau c^m - \frac{\partial c}{\partial t}\big|_{m+1/2}), z_h)$$
$$+ ((Ec\nabla c^{m+1/2} - Ec\nabla c_{m+1/2}), \nabla z_h) + ((u^m \nabla c^{m+1/2} - u_{m+1/2}\nabla c_{m+1/2}), z_h)$$

Hence

$$(\hat{\phi}D_\tau\hat{\xi}^m,\hat{z}_h)+(Ec\nabla\xi^{m+1/2},\nabla z_h)=-(\hat{\phi}D_\tau\hat{\varsigma}^m,\hat{z}_h)-(Ec\nabla\varsigma^{m+1/2},\nabla z_h)-$$

$$((u^m\nabla c^{m+1/2},z_h)-R(U^m,C^{m+1/2},z_h))-((\hat{g}(C^{m+1/2}),\hat{z}_h)-(g(c^{m+1/2}),z_h))-$$

$$(\phi D_\tau c^m,z_h)-(\hat{\phi}D_\tau\hat{c}^m,\hat{z}_h))+((\phi(D_\tau c^m-\frac{\partial c}{\partial t}|_{m+1/2}),z_h)+ \qquad\qquad ...(20)$$

$$((Ec\nabla c^{m+1/2}-Ec\nabla c_{m+1/2}),\nabla z_h)+((u^m\nabla c^{m+1/2}-u_{m+1/2}\nabla c_{m+1/2}),z_h)$$

$$=I1+I2+I3+I4+I5+I6$$

In (20) , let $z_h=\xi^{m+1/2}\in M_h$, and using (A1) the left – hand side is

$$\geq\frac{\hat{\phi}\|\hat{\xi}^{m+1}\|^2-\|\hat{\xi}^m\|^2}{2\tau}+c_0\|\nabla\xi^{m+1/2}\|^2$$

from (A1), we have:

$$I1=(\hat{\phi}D_\tau\hat{\varsigma}^m,\hat{z}_h)\leq M(\|D_\tau\hat{\varsigma}^m\|^2+\|\hat{\xi}^{m+1/2}\|^2)$$

using(7) and (8), we have

$$\|D_\tau\hat{\varsigma}^m\|\leq\|D_\tau\varsigma^m\|+\|D_\tau\varsigma^m-D_\tau\hat{\varsigma}^m\|\leq\|D_\tau\varsigma^m\|+Mh_c|D_\tau\varsigma^m|_1$$

From (9) we have

$$\|\hat{\xi}^{m+1/2}\|\leq M\|\xi^{m+1/2}\|\leq M(\|\xi^{m+1}\|+\|\xi^m\|)$$

$$I1\leq M(\|\xi^{m+1}\|^2+\|\xi^m\|^2+\|D_\tau\varsigma^m\|^2+h_c^2\|D_\tau\varsigma^m\|_1^2)$$

Using (A1), we have: $\qquad I2\leq M\|\nabla\varsigma^{m+1/2}\|^2+\varepsilon\|\nabla\xi^{m+1/2}\|^2$

From lemma (4.2)

$$I3\leq M(h_c^2+\|\xi^{m+1/2}\|^2+\|\varsigma^{m+1/2}\|^2+$$

$$\|\nabla\varsigma^{m+1/2}\|^2+\|u^m-U^m\|^2)+\varepsilon\|\nabla\xi^{m+1/2}\|^2$$

let $\theta=\phi\frac{c^{m+1}-c^m}{\tau}=\phi(\frac{\partial c}{\partial t}|_{m+1/2}+1/24\frac{\partial^3 c}{\partial t^3}|\tau^2)$

using (A1),(A2) and [11], we get

$$I4\leq Mh_c(1+\tau^2)|z_h|_1+Mh_c(1+\tau^2)\|z_h\|$$

$$\leq Mh_c\|z_h\|+Mh_c|z_h|_1+Mh_c\tau^2\|z_h\|+Mh_c\tau^2|z_h|_1$$

$$\leq M(h_c^2+h_c^2\tau^4)+\varepsilon\|\nabla\xi^{m+1/2}\|^2$$

$$I5\leq M(h_c^2+\|\xi^{m+1/2}\|^2+\|\varsigma^{m+1/2}\|^2)+\varepsilon\|\nabla\xi^{m+1/2}\|^2$$

let I6=K1+K2+K3, so

$$K1\leq M(\tau^4+\|\xi^{m+1}\|^2+\|\xi^m\|^2),K2\leq M\tau^2+\varepsilon\|\xi^{m+1/2}\|^2,K3\leq M(\tau^2+\|\xi^{m+1}\|^2+\|\xi^m\|^2)\text{ then}$$

we get

$$I6 \le M(\tau^2 + \left\|\xi^{m+1}\right\|^2 + \left\|\xi^m\right\|^2) + \varepsilon\left\|\nabla\xi^{m+1/2}\right\|^2$$

Equation (20) can be written now

$$\frac{\hat{\phi}\left\|\hat{\xi}^{m+1}\right\|^2 - \left\|\hat{\xi}^m\right\|^2}{2\tau} + c_0\left\|\nabla\xi^{m+1/2}\right\|^2 \le M(\left\|\xi^{m+1}\right\|^2 + \left\|\xi^m\right\|^2 + \left\|D_\tau\varsigma^m\right\|^2 + h_c^2\left\|D_\tau\varsigma^m\right\|_1^2 + \left\|\nabla\varsigma^{m+1}\right\|^2$$
$$+ \left\|\nabla\varsigma^{m+1/2}\right\|^2 + \left\|\varsigma^{m+1/2}\right\|^2 + \left\|u^m - U^m\right\|^2 + h_c^2 + \tau^2) + \varepsilon\left\|\nabla\xi^{m+1/2}\right\|^2$$

Take the summation from 0 to l, where $m \le l \le N_\tau$, and $C^0 = \bar{c}^0$ so $\xi^0 = 0$

$$\hat{\phi}\left\|\hat{\xi}^{l+1}\right\|^2 \le M_1\sum_{m=0}^{l+1} \left\|\xi^m\right\|^2\tau + M_2\left\|D_\tau\varsigma\right\|_{L_\Delta^2(L^2)}^1 + M_2 h_c^2\left\|D_\tau\varsigma\right\|_{L^2(H^1)}^2$$
$$+ M_2\sum_{m=0}^{l}\left\|\varsigma\right\|_1^2\tau + M_2(h_c^2 + \tau^2) + M_3\sum_{m=0}^{l}\left\|u^m - U^m\right\|^2\tau$$

where $\left\|w\right\|_{L_{\Delta(x)}^2}^2 = \sum_{m=0}^{N_\tau}\left\|w^m\right\|_x^2\tau$, using (7) and (A1), we get

$$\left\|\xi^{l+1}\right\| \le \left\|\hat{\xi}^{l+1}\right\| + Mh_c\left|\xi^{l+1}\right|_1 \le \left\|\hat{\xi}^{l+1}\right\| + Mh_c\frac{\sqrt{6}}{k}\left\|\hat{\xi}^{l+1}\right\| \le M\left\|\hat{\xi}^{l+1}\right\| \qquad\qquad …(21)$$

$\hat{\phi}\left\|\hat{\xi}^{l+1}\right\| \ge M_0\left\|\hat{\xi}^{l+1}\right\|$, so

From lemma (4.3) we have

$$\sum_{m=0}^{l}\left\|u^m - U^m\right\|^2\tau \le M(\sum_{m=0}^{l}\left\|\xi^m\right\|^2\tau + \left\|\varsigma\right\|_{L_\Delta^2(L^2)}^2 + h_p^{2r}) \qquad\qquad … (22)$$

From (21) and (22), we have

$$\left\|\xi^{l+1}\right\|^2 \le M_1\sum_{m=0}^{l+1}\left\|\xi^m\right\|^2\tau + M_2(h_c^2 + \tau^2 + h_p^{2r})$$

from Gronwall inequality , we get:

$$\left\|\xi^{l+1}\right\| \le M(h_c + \tau + h_P^r)$$

it is $\left\|c^{l+1} - C^{l+1}\right\| \le M(h_c + \tau + h_P^r)$

Theorem(4.2): For all $m \le l \le N_\tau$ if $\tau \le \tau_0$, then

$$\left\|\xi^{l+1}\right\| + \left\|\pi^{l+1}\right\| + \sum_{m=0}^{l+1}(\left\|\nabla\xi^m\right\| + \left\|\nabla\pi^m\right\|)\tau \le M(\tau + h_c + h_P^r)$$

where M independent of τ , h_c and h_p .

Proof:

We shall begin by deriving an evolution inequality for the difference π between the projection $\bar{p}$ and the approximation solution P. The weak form of eq.(1) is

$$(\phi_1 \frac{\partial p}{\partial t}, v) + (a(c)\nabla p, \nabla v) = (-q, v) + (R_s^{'}, v) \ ; \ \forall v \in H^1(\Omega)$$

so there can be difference between equations (12) and (15) to show that.

$$(\phi_1 D_\tau \pi^m, v) + (a(C^{m+1})\nabla \pi^{m+1}, \nabla v) = -(\{a(c^{m+1}) - a(C^{m+1})\}\nabla \overline{p}^{m+1}, \nabla v) -$$

$$(\phi_1 D_\tau \eta^{m}, v) + \mu(\eta^{m+1}, v) + (\phi_1 (D_\tau p^m - \frac{\partial p}{\partial t}|_{m+1}), v) \qquad \ldots(23)$$

$$= I1 + I2 + I3 + I4$$

Select π^{m+1} as the test function in (23), we have from (17-c) and (A1)

$$I1 \le M \left\| c^{m+1} - C^{m+1} \right\| \left\| \nabla \overline{p}^{m+1} \right\| \left\| \nabla \pi^{m+1} \right\|$$

$$\le M(\left\| c^{m+1} - C^{m+1} \right\| \|\overline{p}\|_{1,\infty} \left\| \nabla \pi^{m+1} \right\|$$

$$\le M(\left\| \varsigma^{m+1} \right\|^2 + \left\| \xi^{m+1} \right\|^2) + \varepsilon^{'} \left\| \nabla \pi^{m+1} \right\|^2$$

From (A1), we have

$$I2 \le M \left\| D_\tau \eta^m \right\| \left\| \pi^{m+1} \right\|$$

$$\le M(\left\| D_\tau \eta^m \right\|^2 + \left\| \pi^{m+1} \right\|^2)$$

and

$$I3 \le M \left\| \eta^{m+1} \right\| \left\| \pi^{m+1} \right\|$$

$$\le M(\left\| \eta^{m+1} \right\|^2 + \left\| \pi^{m+1} \right\|^2)$$

From (A1),(A2) and using [9], we get

$$I4 \le M \left\| D_\tau P^m - \frac{\partial p}{\partial t}(t_{m+1}) \right\| \left\| \pi^{m+1} \right\|$$

$$\le M\tau \left\| \pi^{m+1} \right\|^2 \le M(\tau^2 + \left\| \pi^{m+1} \right\|^2)$$

Now, (23) can be written as

$$\frac{c_0}{\tau}[\left\| \pi^{m+1} \right\|^2 - \left\| \pi^m \right\|^2] + c_0 \left\| \nabla \pi^{m+1} \right\|^2 \le c_0 \left\| D_\tau \pi^{m+1} \right\| \left\| \pi^{m+1} \right\| + \left\| a(C^{m+1}) \right\| \left\| \nabla \pi^{m+1} \right\|^2 \le \qquad \ldots(24)$$

$$M(\left\| \xi^{m+1} \right\|^2 + \left\| \varsigma^{m+1} \right\|^2 + \left\| D_\tau \eta^m \right\|^2 + \left\| \pi^{m+1} \right\|^2 + \left\| \eta^{m+1} \right\|^2 \tau^2) + \varepsilon^{'} \left\| \nabla \pi^{m+1/2} \right\|^2$$

Now by using prove of theorem (4.1) we get

$$\frac{c_0}{2\tau}[\left\| \xi^{m+1} \right\|^2 - \left\| \xi^m \right\|^2] + c_0 \left\| \nabla \xi^{m+1/2} \right\| \le M(\left\| \xi^{m+1} \right\|^2 + \left\| \xi^m \right\|^2 + \left\| D_\tau \varsigma^m \right\|^2$$

$$+ h_c^2 \left\| D_\tau \varsigma^m \right\|_1^2 + \left\| \nabla \varsigma^{m+1} \right\|^2 + \left\| \nabla \varsigma^{m+1/2} \right\|^2 + \left\| \varsigma^{m+1/2} \right\|^2 \qquad \ldots(25)$$

$$+ \left\| u^m - U^m \right\|^2 + h_c^2 + \tau^2) + \varepsilon^{'} \left\| \nabla \xi^{m+1/2} \right\|^2$$

Combine equation (24) and (25)and where $\varepsilon' > 0$ is arbitrary small constant, move $\dfrac{\varepsilon'}{2}\left\|\nabla\xi^{m+}\right\|^2$ into left side and take $\varepsilon' > 0$ small enough such that $c_0 - \varepsilon' > 0$

$$\frac{c_0}{\tau}[\left\|\pi^{m+1}\right\|^2 - \left\|\pi^m\right\|^2] + \frac{c_0}{2\tau}[\left\|\xi^{m+1}\right\|^2 - \left\|\xi^m\right\|^2] + c_0\left\|\nabla\pi^{m+1}\right\|^2 + \frac{c_0}{2}\left\|\nabla\xi^{m+1/2}\right\|^2 \le$$

$$M(\left\|\xi^{m+1}\right\|^2 + \left\|\xi^m\right\|^2 + \left\|\pi^{m+1}\right\|^2 + \left\|\varsigma^{m+1}\right\|_1^2 + \left\|D_\tau\varsigma^m\right\|^2$$

$$+ h_c^2\left\|D_\tau\varsigma^m\right\|_1^2 + \left\|\varsigma^{m+1/2}\right\|_1^2 + \left\|D_\tau\eta^m\right\|^2 + \left\|\eta^{m+1}\right\|^2 + \left\|u^m - U^m\right\|^2 + h_c^2 + \tau^2)$$

$$+ \varepsilon'(\left\|\nabla\pi^{m+1}\right\|^2 + \frac{\left\|\nabla\xi^{m+1}\right\|^2}{2})$$

then

$$[\left\|\pi^{m+1}\right\|^2 - \left\|\pi^m\right\|^2] + [\left\|\xi^{m+1}\right\|^2 - \left\|\xi^m\right\|^2] + \left\|\nabla\pi^{m+1}\right\|^2\tau + \left\|\nabla\xi^{m+1/2}\right\|^2\tau \le$$

$$M_1(\left\|\xi^{m+1}\right\|^2 + \left\|\pi^{m+1}\right\|^2)\tau + M_2(\left\|\varsigma^{m+1}\right\|_1^2 + \left\|D_\tau\varsigma^m\right\|^2$$

$$+ h_c^2\left\|D_\tau\varsigma^m\right\|_1^2 + \left\|\varsigma^{m+1/2}\right\|_1^2 + \left\|D_\tau\eta^m\right\|^2 + \left\|\eta^{m+1}\right\|^2 + \left\|u^m - U^m\right\|^2 + h_c^2 + \tau^2)\tau$$

$$+ \varepsilon(\left\|\nabla\pi^{m+1}\right\|^2 + \left\|\nabla\xi^{m+1}\right\|^2)\tau$$

Take the summation from 0 to l, where

$m \le l \le N_\tau$, and $C^O = \bar{c}^0$ so $\xi^0 = 0$, $P^0 = \bar{p}^0$ so $\pi^0 = 0$

$$\left\|\xi^{l+1}\right\|^2 + \left\|\pi^{l+1}\right\|^2 + \sum_{m=0}^{l+1}\left\|\nabla\xi^m\right\|^2\tau + \sum_{m=0}^{l+1}\left\|\nabla\pi^m\right\|^2\tau \le$$

$$M_1\sum_{m=0}^{l+1}(\left\|\xi^m\right\|^2 + \left\|\pi^m\right\|^2)\tau + M_2(\left\|u^m - U^m\right\|^2\tau + \left\|\varsigma\right\|_{L_\Delta^2(H^1)}^2 + \left\|D_\tau\varsigma\right\|_{L_\Delta^2(L^2)}^2$$

$$+ h_c^2\left\|D_\tau\varsigma\right\|_{L_\Delta^2(H^1)}^2 + \left\|\varsigma\right\|_{L_\Delta^2(H^1)}^2 + \left\|\eta\right\|_{L_\Delta^2(L^2)}^2 + \left\|D_\tau\eta\right\|_{L_\Delta^2(L^2)}^2 + h_c^2 + \tau^2)$$

$$+ \varepsilon\sum_{m=0}^{l+1}(\left\|\nabla\pi^m\right\|^2 + \left\|\nabla\xi^m\right\|^2)\tau$$ where

$\left\|w\right\|_{L_\Delta^2(x)}^2 = \sum_{m=0}^{N_\tau}\left\|w^m\right\|_x^2\tau$, from lemma (4.3), we have

$$\sum_{m=0}^{l}\left\|u^m - U^m\right\|^2\tau \le k_3(\sum_{m=0}^{l}\left\|c^m - C^m\right\|^2\tau + h_p^{2r}) \le M_3(\sum_{m=0}^{l}\left\|\xi^m\right\|^2 + \left\|\varsigma\right\|_{L_\Delta^2(L^2)}^2 + h_p^{2r})$$ and using (17),(18) we get

$$\left\|\xi^{l+1}\right\|^2 + \left\|\pi^{l+1}\right\|^2 + \sum_{m=0}^{l+1} \left\|\nabla \xi^m\right\|^2 \tau + \sum_{m=0}^{l+1} \left\|\nabla \pi^{\ m}\right\|^2 \tau \leq$$

$$M_1 \sum_{m=0}^{l+1} (\left\|\xi^{\ m}\right\|^2 + \left\|\pi^m\right\|^2)\tau + M_4(h_c^2 + h_p^{2r} + \tau^2)$$

$$+ \varepsilon \sum_{m=0}^{l+1} (\left\|\nabla \pi^{\ m}\right\|^2 + \left\|\nabla \xi^m\right\|^2)\tau$$

From Gronwall inequality, for τ *and* ε small enough, we get

$$\left\|\xi^{l+1}\right\| + \left\|\pi^{l+1}\right\| + \sum_{m=0}^{l+1} \left\|\nabla \xi^m\right\|\tau + \sum_{m=0}^{l+1} \left\|\nabla \pi^{\ m}\right\|\tau \leq M(h_c + h_p^r + \tau)$$

where M independent of τ, h_c *and* h_p.

Theorem(4.5): with the assumption (A1)~ (A3), if $h_c = O(h_P), \tau = O(h_P)$, then

$$\left\|c-C\right\|_{L^\infty(l^2)} + \left\|p-P\right\|_{L^\infty(l^2)} + \left\|u-U\right\|_{L^\infty(l^2)} + \left\|c-C\right\|_{L^\infty(H^1)} + \left\|p-P\right\|_{L^\infty(H^1)} \leq M(\tau + h_c + h_P^r)$$

Proof:

With $c - C = \varsigma + \xi$ *and* $p - P = \eta + \pi$ we have

$$\left\|c-C\right\|_{L^\infty(l^2)} \leq \left\|\varsigma\right\|_{L^\infty} + \left\|\xi\right\|_{L^\infty}$$

$$\left\|p-P\right\|_{L^\infty(l^2)} \leq \left\|\eta\right\|_{L^\infty} + \left\|\pi\right\|_{L^\infty}$$

From lemma(4.3) and theorem (4.1)

$$\left\|u-U\right\|_{L^\infty(l^2)} \leq M(\tau + h_c + h_P^r)$$

By using (Raviart,1979), we get

$$\left\|c-C\right\|_{L^\infty(H^1)} \leq \left\|\varsigma\right\|_1 + \left\|\xi\right\|_1 \leq M(\left\|\varsigma\right\|_1 + \left\|\nabla \xi\right\|) \ and$$

$$\left\|p-P\right\|_{L^\infty(H^1)} \leq M(\left\|\eta\right\|_1 + \left\|\nabla \pi\right\|)$$

then by using theorem (4.2) and eq. (17) we get

$$\left\|c-C\right\|_{L^\infty(l^2)} + \left\|p-P\right\|_{L^\infty(l^2)} + \left\|u-U\right\|_{L^\infty(l^2)} + \left\|c-C\right\|_{L^\infty(H^1)} + \left\|p-P\right\|_{L^\infty(H^1)} \leq M(\tau + h_c + h_P^r)$$

which complete the proof.

5. Numerical Application

In this example, we solve a purely convective problem in one dimension [12]

$$-u_y \frac{\partial c}{\partial y} = \frac{\partial c}{\partial t}$$

Where c is the concentration in the region $-2 \leq y \leq 0$, subject to the boundary conditions

$$c = 1 \qquad y = 0 \qquad 0 \le t \le 0.2$$

$$c = 0 \qquad y = 0 \qquad t > 0.2$$

$$\frac{\partial c}{\partial y} = 0 \qquad y = -2 \qquad \textit{for all } t$$

we apply two methods: Galerkin and a kind of partial upwind finite element for this example. We discrete the region $-2 \le y \le 0$ into at first 100 quadrilateral elements and second 200 triangular elements with 202 nodes, also the distance between any two nodes is 0.02 and take $u_y = 1.0$, theta $= 0.5$, $\tau = 0.04$, and the number of steps (N) = 25 on a 1*100 mesh for quadrilateral element and 1*200 mesh for triangular element.

The correct solution to the problem is described by a rectangular pulse moving with unit velocity in the y direction. Table (1) contains numerical results where $\theta = 1/2$ at all nodes lie on the right-hand side of the finite element mesh in this example. From the boundary conditions we can see that the solution at nodes 1 and 2 is held at the value 1.0 for the first 0.2 seconds of convection. Figure (1) shows the computed solution after one second and it draws between the concentration and coordinate y, we can see that while y convergence to zero the value of concentration convergence oscillation to solution.

Table (1): The numerical solutions at all nodes of the right - hand side of mesh, where theta = 0.5, $\tau = 0.04$, N = 25

| Coordinate Y	Galerkin method (Quadrilateral Element)	Galerkin method (Triangular Element)	Kind of Partial Upwind F.E.M.
0.0000E+00	0.3289E-24	-0.3987E-24	-0.3987E-24
-0.2000E-01	-0.4934E-02	0.4238E-02	0.4238E-02
-0.4000E-01	0.1548E-01	-0.1405E-01	-0.1405E-01
-0.6000E-01	-0.4225E-02	0.2256E-01	0.2256E-01
-0.8000E-01	-0.2975E-01	-0.7050E-02	-0.7051E-02
-0.1000E+00	0.3345E-01	-0.2730E-01	-0.2730E-01
-0.1200E+00	0.2570E-01	0.5020E-01	0.5020E-01
-0.1400E+00	-0.6509E-01	-0.2636E-01	-0.2636E-01
-0.1600E+00	-0.1126E-01	-0.5949E-01	-0.5949E-01
-0.1800E+00	0.9229E-01	0.8407E-01	0.8406E-01
-0.2000E+00	0.4207E-02	0.5155E-01	0.5155E-01
-0.2200E+00	-0.1170E+00	-0.1160E+00	-0.1160E+00
-0.2400E+00	-0.2221E-01	-0.5481E-01	-0.5481E-01
-0.2600E+00	0.1329E+00	0.1267E+00	0.1267E+00
-0.2800E+00	0.7632E-01	0.8523E-01	0.8523E-01
-0.3000E+00	-0.1111E+00	-0.1125E+00	-0.1125E+00
-0.3200E+00	-0.1485E+00	-0.1465E+00	-0.1465E+00
-0.3400E+00	0.1559E-01	0.3309E-01	0.3309E-01
-0.3600E+00	0.1643E+00	0.1784E+00	0.1784E+00
-0.3800E+00	0.1244E+00	0.1156E+00	0.1156E+00
-0.4000E+00	-0.4353E-01	-0.6673E-01	-0.6673E-01
-0.4200E+00	-0.1594E+00	-0.1742E+00	-0.1742E+00
.	.	.	.
.	.	.	.
.	.	.	.
-0.1860E+01	0.6962E-11	0.4492E-10	0.4492E-10
-0.1880E+01	0.3435E-11	0.2433E-10	0.2427E-10
-0.1900E+01	0.1688E-11	0.1313E-10	0.1320E-10
-0.1920E+01	0.8274E-12	0.7031E-11	0.7033E-11
-0.1940E+01	0.4025E-12	0.3846E-11	0.3703E-11
-0.1960E+01	0.1974E-12	0.2032E-11	0.2236E-11
-0.1980E+01	0.9360E-13	0.1000E-11	0.9707E-12
-0.2000E+01	0.4779E-13	0.7114E-12	0.3765E-12
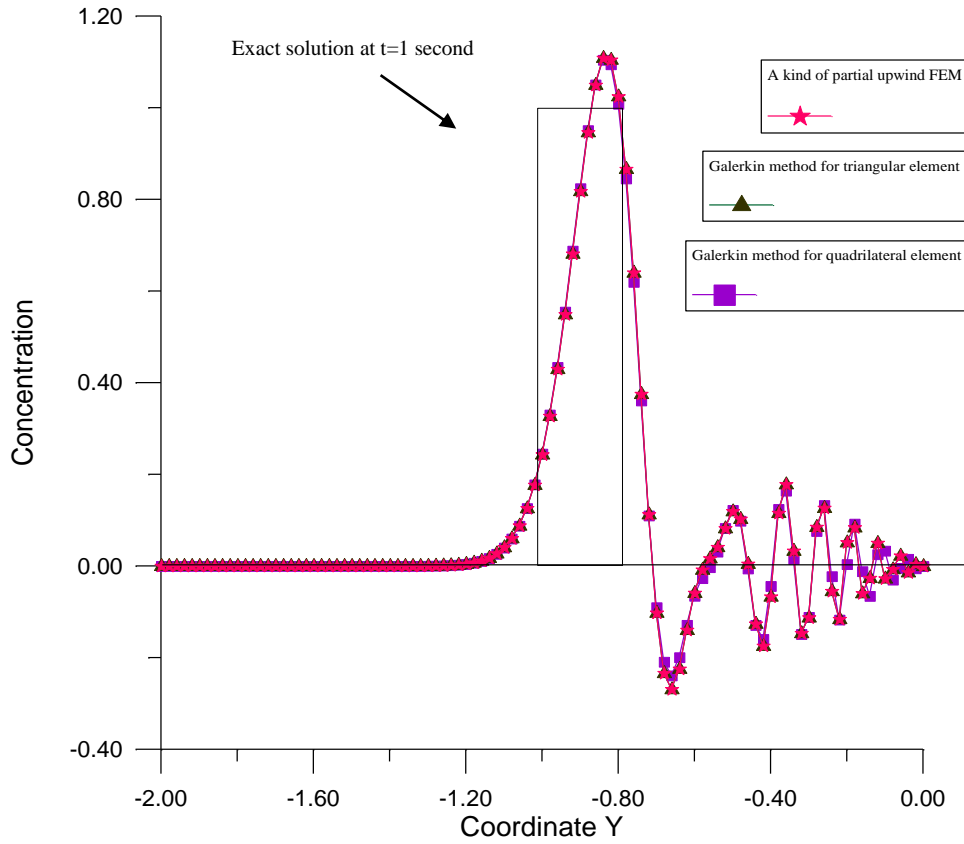
Figure (1): The solutions in example after one second
for theta =0.5, τ =0.04 , N =25

6. Conclusions

We used the system with large coupled of strongly non-linear partial differential equations which arise from the contamination of nuclear waste in porous media .We used a Galerkin method for the pressure equation and a kind of partial upwind finite element method for the concentration.For the compressible case, we obtained the error estimates for approximate Darcy velocity U, concentrations   C in $L^{\infty}(0,\tilde{T},L^{2}(\Omega))$ .From the numerical results presented in this application, we have got a kind of partial upwind finite element method for triangular element convergent to the exact solution and in comparison with Galerkin method , we found that a kind of

partial upwind finite element method much more accurate than Galerkin method.

NOTATIONS

c Concentration of the brine in fluid

c_p Specific heat

c_R Compressibility of rock formation

c_T Coefficient of thermal expansion

c_w Compressibility of the fluid

D   Dispersion tensor

D_m Molecular diffusion

E   Dispersivity tensor (hydrodynamic + molecular)

k   Permeability tensor

N   Number of nuclei

p   Pressure

q   Rate of fluid withdrawal

R_s Brine source rate due to salt dissolution

R'_s Fluid source rate due to salt dissolution

u   Darcy velocity vector

ϕ Porosity

ϕ_0 Porosity at the reference pressure

μ Viscosity

R   Rock (formation)

w   Water (fluid)

s   Salt (brine)

τ Time step

W_s^k Sobolev space

e   closed triangle element in T_h

h_c Step length for the concentration

h_p Step length for the pressure

U   Approximate velocity

P   Approximate pressure

C   Approximate concentration

$\bar{u}$ Projection of velocity

$\bar{p}$ Projection of pressure

$\bar{c}$ Projection of concentration

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