On CS- Rings

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ABSTRACT

The main purpose of this paper is to study CS-rings. We give some properties of right CS-rings and the connection between such rings and reduced rings, regular rings, strongly regular rings, and S-weakly regular rings.

Keywords: Reduced Rings, Regular Rings, Strongly Regular Rings

حول حلقات من النمط CS

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الملخص

الهدف الرئيسي من هذا البحث هو دراسة الحلقات من النمط CS. إذ أعطينا بعض خواص الحلقات من النمط CS اليمنى ودرسنا العلاقة بين هذا النوع من الحلقات والحلقات المختزلة والحلقات المنتظمة والحلقات المنتظمة بقوة ، والحلقات المنتظمة بضعف من النمط S .

الكلمات المفتاحية : الحلقات المختزلة ، الحلقات المنتظمة ، الحلقات المنتظمة بقوة

1.Introduction:

Throughout this paper, R represents associative rings with identity. A ring R is called CS-ring (or extending ring) if every right ideal is essentially contained in a direct summand of R[4], The word CS-ring means "Complements are Summands" equivalently definition of every complement right ideal is a direct summand.

Y(R) will denote the **right singular ideal** of R. N(R) is **nil radical** of R. For every $a \in R$, r(a) and l(a) will stand respectively for **right** and **left annihilators** of a. An element $0 \neq a \in R$ is said to be **left regular** if l(a)=0. A ring R is said to be **NI** if N(R) forms an ideal of R[7]. An idempotent element $e \in R$ is called left (resp. right) **semicentral** if xe=exe (resp. ex=exe), for all $x \in R$ [1]. A right ideal I of a ring R is **closed** if there is no right ideal of R which is a proper essential extension of I [6], clearly every maximal right ideal is right closed.

2. **CS-ring (Basic properties):**

Following[4], a ring R is said to be right (left) CS-ring if every non-zero right (left) ideal is essential in a direct summand, equivalently, every right (left) closed ideal is a direct summand[6].

Example: Let R={ $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, a,b,c \in Z_3$ }(Z3 is the ring of integers modulo 3)

It can be checked that R is CS-ring.

Proposition 2.1: Let R be a right CS-ring, then for every right ideal I of R,

1- $l(I) \neq 0$.
2- There exists an idempotent element $0 \neq e \in R$ such that eI=I, if $e \in I$, then $I^2=I$.

Proof(1):

Let I be non zero right ideal of R such that I is an essential in a direct summand, then there exist right ideals J , K of R such that $I \subseteq J$ and $J \oplus K$ = R. In particular j+k=1, $j \in J$ and $k \in K$, thus jx+kx=x, for all $x \in I$, kx=x-jx, since $x \in I \subseteq J$ and $jx \in J$(since J is a right ideal), then $x-jx \in J$, follows that $kx \in J$ and $kx \in K$(since K is a right ideal), $kx \in J \cap K=0$. Therefore kx=0 which implies that kI=0. Hence $k \in l(I)$, thus l(I) $\neq 0$.

Proof(2):

Let I be non zero right ideal of R such that I is an essential in a direct summand, then there exist right ideals J, K of R such that $I \subseteq J$ and $J \oplus K$ = R. In particular e+k=1,where $e \in J$, $k \in K$, then $e^2+ke=e$, $ke \in J \cap K=0$ implies $e^2=e$, now e+k=1, $e \in J$ and $k \in K$, thus ey + ky = y, ey=y for all $y \in I$, therfore eI=I.

Let $e \in I$,we shall to prove $I^2=I$. It is clearly that $I^2 \subseteq I$, since ey=y for all $y \in I$, now $e,y \in I$ and $ey \in I^2$, therefore $I \subseteq I^2$, hence $I^2=I$.

Lemma 2.2: Let R be a right CS-ring, then for every closed right ideal $I \neq 0$:

1- eR=I, e is an idempotent element.
2- I is a left pure ideal.

Proof(1):

Let $0 \neq I$ be a right closed ideal, by (Proposition 2.1(2)) there exists an idempotent element $e \in I$ such that $I = eI \subseteq eR$, since $e \in I$,we have eR $\subseteq$ I, therefore eR = I.

Proof(2):

Since I is closed right ideal, then there exists right ideal J in R such that $I \oplus J=R$. In particular x+j=1, $x \in I$ and $j \in J$, xi+ji = i , ji = 0, therefore xi=i, hence I is a left pure ideal.

Proposition 2.3: Let R be right CS-ring and every maximal right ideal is genrated by right semicentral element, then r(a)+r(b)=R for any $a,b \in R$ such that ab=0.

Proof:

Asume that $r(a)+r(b) \neq R$ and let M be a maximal right ideal containing r(a)+r(b). Since ab=0, implies that $b \in r(a) \subseteq M$, there exist $e \in M$ such that b=eb. eb=ebe=be by (Lemma 2.2(2) and e is a right semicentral), so b=be and b(1-e)=0 implies $l\text{-}e \in r(b) \subseteq M$, $1 \in M$, a contradiction, therefore r(a)+r(b)=R.

Theorem 2.4: Let R be a right CS-ring, then Y(R)=0.

Proof:

Let $Y(R) \neq 0$, there exist $0 \neq a \in Y(R)$, such that $a^2=0$ implies $a \in r(a)$. Since R is a right CS-ring then there exists $0 \neq J$ right ideal in R is direct summand $r(a) \subseteq J$ such that $J \oplus K=R$ (K is a right ideal of R) but $r(a) \subseteq J$ implies $r(a) \cap K=0$, since r(a) is essential, a contradiction, therefore Y(R)=0.

3.The connection between CS-ring and other rings:

In this section we study the connection between CS-rings and strongly regular, S-weakly regular and reduced rings.

Following[5] a ring R is called **strongly right bounded (**briefly **SRB)** if every non-zero right ideal contains a nonzero two-sided ideal of R. A ring is called **Strongly regular ring** if for all $a \in R$ there exists $b \in R$ such that $a=a^2b$.

Theorem 3.1: Let R be a right CS-ring , then R is a strongly regular ring if one of the following holds:

1- for all $a \in R$, $l(a) \subseteq r(a)$.
2- R is quasi-duo ring.

Proof(1):

Let $a \in R$ such that aR+r(a)=R. If not, there exists a maximal right ideal M containing aR+r(a), by (Lemma 2.2(2)) M is a left pure ideal. Since $a \in M$ there exists $b \in M$ such that a=ba, then $1\text{-}b \in l(a) \subseteq r(a) \subseteq M$, then $1 \in M$, a contradiction, in particular ar+t=1, for some $r \in R$ and $t \in r(a)$, where $a^2r=a$, this proves that R is a strongly regular ring.

Proof(2):

Assume Ra+l(a)$\neq$R and a $\in$ R. Let M be a maximal ideal in R containing Ra+l(a). By (Lemma 2.2(1)) there exists b$\in$M such that a=ba. Then (1-b)a=0 implies 1-b$\in$l(a)$\subseteq$M, 1$\in$M, a contradiction, hence Ra+l(a)=R. In particular ra+t=1, for some r$\in$R and t$\in$l(a), where ra^2=a, this proves that R is a strongly regular ring.

Corollary 3.2: Let R be right CS-ring, then R is a quasi-duo ring if and only if R is a reduced ring.

Proposition 3.3: If R is SRB and right CS-rings, then R is a reduced ring.
Proof:

Let a^2=0 such that 0$\neq$a$\in$R, a $\in$ r(a), then by (Theorem 2.4) Y(R)=0. Thus there exists a right ideal J such that r(a)$\cap$J=0. Since R is SRB-ring, then there exists 0$\neq$I a two sided ideal in J, since aI$\subseteq$aR, aR$\subseteq$r(a) (since a$\in$r(a)) implies aI$\subseteq$r(a) and aI$\subseteq$I$\subseteq$J. Therefore aI$\subseteq$r(a)$\bigcap$J=0 implies that aI=0, then I$\subseteq$r(a), thus I$\subseteq$r(a)$\bigcap$J=0, I=0, a contradiction, hence a=0, R is a reduced ring.

Corollary 3.4: If R is SRB and right CS-rings, then R is a strongly regular ring.
Proof:

By (Proposition 3.3) and (Theorem 3.1(1)).

Recall that, R is **S-weakly regular** ring if for all a$\in$R ,a$\in$Ra^2Ra [3]. A ring R is called **ZI** if for a,b$\in$R, ab=0 implies aRb=0 [7].

Lemma 3.5: Let R be a ZI and right CS-ring, then R is an S-weakly regular ring.
Proof:

Let a$\in$R such that Ra^2R+l(a)=R. If not, there exists a maximal right ideal M containing Ra^2R+l(a), since M is a direct summand there exists K right ideal such that M$\oplus$K=R, that is meaning Ra^2R+l(a)$\cap$K=0, implies that K$a^2$$\subseteq$R$a^2R\cap$K=0. Therefore K$a^2$=0 implies Ka$\subseteq$l(a). If Ka=0 implies that K$\subseteq$l(a)$\subseteq$M but M$\cap$K=0, a contradiction. If Ka$\neq$0, Ka$\subseteq$M$\cap$K=0 implies that Ka=0, a contradiction, hence Ra^2R+l(a)=R, in particular ra^2s+t=1, where r,s$\in$R and t$\in$l(a), so ra^2sa+ta=a, for all a$\in$R , ta=0 implies that ra^2sa=a, then a$\in$Ra^2Ra, therefore R is an S-weakly regular ring.

Theorem 3.6: Let R be a right CS-ring , then R is a reduced ring if every idempotent elements are left semicentral .
Proof:

Let a^2=0 such that a$\neq$0, a$\in$r(a), then there exists a maximal right ideal M containing r(a). Since M is a direct summand, by (Lemma 2.2(1)) there exists an idempotent element e in R such that I=eR. (1-e)e=0, which implies that $eR \subseteq r(1\text{-}e)$. Then $M \subseteq r(1\text{-}e)$, now let $x \in r(1\text{-}e)$, then (1-e)x=0, then x-ex=0, then x=ex. Since $e \in M$, then $ex \in M$, $x \in M$. Then $r(1\text{-}e) \subseteq M$, therefore M=r(1-e).

Since $a \in M$, then (1-e)a=0 implies that (1-e)a(1-e)=0 (since (1-e) is left semicentral). Then (1-e)a(1-e)=a(1-e)=0, that is meaning $(1\text{-}e) \in r(a) \subseteq M = r(1\text{-}e)$, $(1\text{-}e)^2 = 0$, a contradiction, therefore a = 0 and R is reduced ring.

Corollary 3.7: Let R be a right CS-ring, then R is a strongly regular ring if every idempotent elements are left semicentral .

Proof:

By (Theorem 3.6) and (Theorem 3.1(1)) .

Lemma 3.8: Let R be a right CS-ring, then for every primitive idempotent element e of R , eR is a uniform right ideal of R.

Proof: [2 ,Lemma(1)].

Lemma 3.9: Let R be a right CS-ring . Then R is a regular ring, if every maximal ideal generated by a primitive idempotent element.

Proof:

Let $0 \neq a \in R$**,** then aR is a right ideal which is essential in a direct summand. Then there exist right ideals J and K in R such that $J \oplus K = R$ and $aR \subseteq J$.

Then $aR \cap K = 0$, if $aR \oplus K \neq R$, then there exists a maximal right ideal M such that $aR + K \subseteq M$, by (Lemma 3.8) M is a uniform right ideal of R, but $aR \cap K = 0$. Contradiction, then aR+K=R, and $aR \cap K = 0$, then there exist r in R, k in K and 1 in R. Such that ar+k=1, ara + ka = a, $ka \in aR \cap K = 0$, then ka = 0, then ara = a , therefore R is a regular ring.

Lemma 3.10: Let R be a right CS-ring. Then every left regular element is right invertible.

Proof:

Let $0 \neq a \in R$ be a left regular element, and $aR \neq R$, aR is a right ideal in a right CS-ring, therefore there exists a non-zero right ideals I and J such that $I \oplus J = R$ and $aR \subseteq I$.

Now $jar \in I \cap J = 0$ implies that jar=0, therefore $j \in l(ar) \neq 0$ for all $r \in R$. In particular take r=1. We have $l(a) \neq 0$, a contradiction. Therefore aR=R, then a is a right invertible element.

Lemma 3.11: Let R be a right CS-ring. Then R is a simple ring if 0 and 1 are the only idempotent elements in R.

Proof:

Let $0 \neq I$ be a right ideal of R, since R is a right CS-ring, then there exist right ideals J and K of R such that $J \oplus K=R$ and $I \subseteq J$. In particular e+k=1, $e \in J$ and $k \in K$, we have e and k are an idempotent elements in R. Therefore e=0 and k=1 or e=1 and k=0. If e=0 and k=1 by (Lemma 2.1(2)) eI=I implies eI=0, then I=0, a contradiction. It must to be e=1 and k=0 by (Lemma 2.1(1)) $k \in L(I) \neq 0$ and $k \neq 0$, a contradiction. Therefore J is not a direct summand containing a right ideal I, but R is a right CS-ring. Therefore I must be a direct summand, similarly to above, we have a contradiction. Hence I=0, then R is a simple ring.

Theorem 3.12: If R is a NI and right CS-ring, then either R is a simple ring or R contains a reduced right ideal (If the reduced right ideal is a maximal then R is a reduced ring)

Proof:

Since R is a NI-ring, then either N(R)=R , since every element is nilpotent except 0 and 1 implies that 0 and 1 is the only idempotent element in R, by(Lemma 3.9) R is simple ring, or $N(R) \neq R$, then if N(R)=0, then R is reduced ring, now let $N(R) \neq 0$, since R is a right CS-ring, then N(R) is essential in a direct summand right ideal J, therefore there exists K right ideal such that $J \oplus K=R$ implies $N(R) \cap K=0$, hence K is a reduced right ideal.

If the reduced right ideal is a maximal, since N(R) is a nil ideal, then $N(R) \subseteq J(R)$ implies that $N(R) \subseteq K$, since K is a maximal right ideal, but K is a reduced right ideal, a contradiction, then N(R)=0, therefore R is a reduced ring .

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