Stability Analysis of Gray-Scott Model in One-dimension

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ABSTRACT

In this Paper, we studied the stability analysis of steady state solutions of Gray-Scott Model in one-dimension using Fourier mode and we showed that the solutions are conditionally stable.

Keywords: Gray-Scott Model, stability analysis**.**

تحليل الاستقرارية لأنموذج گراي–سكوت في بعد واحد

عباس يونس البياتي   سعد عبدالله مناع   عبدالغفور محمد امين الروژبياني

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تاريخ استلام البحث:2007/03/27   تاريخ قبول البحث:2007/06/28

الملخص

في هذا البحث درسنا الاستقرارية التحليلية للحلول اللازمنية لنموذج Gray–Scott في البعد الواحد باستخدام تحليل الأستقرارية من النمط فورير, وتبين لدينا أن الحلول مستقرة على نحو مشروط.

الكلمات المفتاحية: أنموذج گراي-سكوت, تحليل الاستقرارية.

1. Introduction:

A system of nature whatever that exists in stable state, in one sense or another, if small disturbance or change in the system, does not exist in time-dependent state in which the planets move about the sun in an orderly fashion. It is known that small additional celestial body is introduced into the system, and then the original state is stable to small disturbance. Similar equations of stability arise in every physical problem [5].

Saad and Bashar, studied the stability of a model of a fully developed laminar fluid flow in rectangular bend duct with secondary flow has been disturbed [9].

Sherratt [12] derived a condition for the wave train itself to be a stable solution, and present numerical evidence for a complex sequence of bifurcations in the unstable region of parameter space in reaction-diffusion equations.

Wan et. al [13] is concerned with several eigenvalue problems in the linear stability analysis of steady state morphogen gradients for several

models of Drosophila wing imaginal discs including on not previously considered.

Gray and Scott [1] studied the simplest case, uniform temperatures and concentrations in the isothermal, and the simplest of the reaction schemes: (i) quadratic autocatalysis ($A + B \rightarrow 2B$); and (ii) cubic autocatalysis ($A + 2B \rightarrow 3B$). The catalyst B may be stable or have a finite lifetime. ($B \rightarrow inert$ products). Allowing for this finite lifetime adds another dimension to our interest.

Scott [10] have considered the autocatalytic reactions: $A \rightarrow B$; rate $= \alpha \, uv^n$, n=0,1 or 2 where u and v are the concentrations of A and B, respectively. Interest centered mainly on irreversible system but for which the catalytic species not indefinitely stable, decaying instead by a rate proportional to its concentration r. In practice all chemical reactions are, to some extent, reversible. The present work investigates the effect of reversibility for the cases in which B does not decay.

Then Gray and Scott [2] studied the cubic autocatalytic reaction ($A + 2B \rightarrow 3B$) forms the basis for the simplest homogenous system to display "exotic" behavior, even under well-stirred, isothermal, open conditions (CSTR). They find multi stability, hysterias, extinction and anomalous relaxation times.

Scott [11] showed that the reactions that representations of non-linear chemical feedback in an isothermal system are the proper type autocatalytic steps

quadratic $u + v \rightarrow 2v$ rat=k_1 uv

cubic $u + 2v \rightarrow 3v$ rat=k_1 uv^2
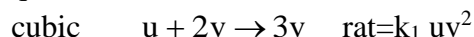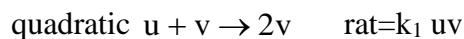
are coupled with the diffusion of the reactants through a permeable boundary form and an external reservoir when the concentrations are held constant.

In this paper, we study the steady state solution and disturbance cases, when the wave amplitudes are constants of the system parameters. The Gray-Scott scheme, which presents cubic-autocatalysis with linear catalyst decay, has been much considered, because of its multiple steady-state response and oscillatory solutions for review and descriptions of much of this work. The scheme is

u+2v $\rightarrow$ 3v, rate = βuv^2, v $\rightarrow$ w, rate =$\beta\gamma$v,   …(1)
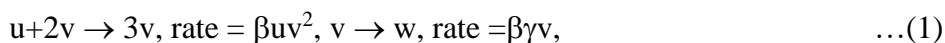
Where the concentrations of the reactant and autocatalyst are u and v, respectively. The parameters β and γ are rate constants. The catalyst is not stable, but undergoes a simple linear decay to a product w. This allows a much wider variety of behavior in the system, than does the cubic reaction alone ([3], [4]). The nonlinear phenomena may be due to feedback through

the detailed chemical mechanism or through departure from the isothermal state [1].

2. The Mathematical Model

The cubic-autocatalytic reaction with linear decay (1) is considered in a non dimensional reaction-diffusion cell are as follows:

$$u_t = u_{xx} - \beta uv^2, \qquad\qquad\qquad (2a)$$
$$v_t = v_{xx} + \beta uv^2 - \beta\gamma\, v \qquad\qquad (2b)$$
$$u_x = v_x = 0 \ \text{ at } \ x = 0, \qquad\quad (2c)$$
$$u = 1, v = v_0, \ \text{ at } x = 1 \text{ and } t = 0 \qquad (2d)$$

$$\dots(2)$$

The system (2) is in non-dimensional form with the concentrations of the reactant and autocatalyst given by u and v, respectively. The reactor has a permeable boundary at x = 1. Joined to a reservoir which contains u and v at constant concentrations. The boundary condition at x = 0 is a symmetry condition an identical reservoir is located at x = -1. The system is characterized by three non-dimensional parameters. The ratio of the autocatalyst and reactant concentrations in the reservoir is v_0. The parameter β is a measure of the importance of the reaction terms, compared with diffusion, while γ is a measure of the importance of autocatalyst decay, compared with the cubic-reaction. The simplest way to adjust the non-dimensional parameters experimentally is by changing the reservoir concentrations.

Other possibilities for varying the non-dimensional parameters include changing the diffusivity of the system or the length of the reactor. The diffusivity could be changed by adjusting the temperature or by the addition of otherwise inactive salts [7].

3. Stability of the model:

We study the analysis of non-dimensional Gray-Scott model in one dimension (2) using Fourier mode, and we assume that the value of concentrations of the reactant u and autocalyst v, has the following form [5]:

$$u(x,t) = u_1(x) + u_2(x,t)$$
$$v(x,t) = v_1(x) + v_2(x,t)$$

$$\dots (3)$$

where u_1 and v_1 denote the steady state case and u_2 and v_2 denote the disturbance case.

If we substitute (3) in (2) and neglecting the nonlinear terms, we get the following two systems (steady state system):

$$\frac{\partial^2 u_1}{\partial x^2} - \beta u_1 v_1^{\,2} = 0 \qquad\qquad (4a)$$

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$$\frac{\partial^2 v_1}{\partial x^2} - \beta v_1 = 0 \qquad \text{(4b)}$$

$$\frac{\partial u_1}{\partial x} = \frac{\partial v_1}{\partial x} = 0 \quad \text{at} \quad x = 0 \qquad \text{(4c)}$$

$$u_1 = 1, v_1 = v_o \quad \text{at} \quad x = 1 \text{ and } t = 0 \qquad \text{(4d)}$$

...(4)

and the disturbance system has the form:

$$\frac{\partial u_2}{\partial t} = \frac{\partial^2 u_2}{\partial x^2} - \beta[2u_1 v_1 v_2 + u_2 v_1^2] \qquad \text{(5a)}$$

$$\frac{\partial v_2}{\partial t} = \frac{\partial^2 v_2}{\partial x^2} + \beta[2u_1 v_1 v_2 + u_2 v_1^2] - \beta\gamma v_2 \qquad \text{(5b)}$$

$$\frac{\partial u_2}{\partial x} = \frac{\partial v_2}{\partial x} = 0 \quad \text{at} \quad X = 0 \qquad \text{(5c)}$$

$$u_2 = 1, v_2 = v_o \quad \text{at} \quad x = 1 \text{ and } t = 0 \qquad \text{(5d)}$$

...(5)

4. Steady state case solutions:

For treatment of stability of the model, first the whole solution of the steady state case , we solve equation (4b) for v_1 we get:

$$v_1(x) = \frac{v_0}{S}\left[e^{\sqrt{\beta\gamma}\, x} + e^{-\sqrt{\beta\gamma}\, x}\right],$$

... (6)

where $\quad S = \left[e^{\sqrt{\beta\gamma}} + e^{-\sqrt{\beta\gamma}}\right].$

For the boundary condition (4d) we get:

$$v_1(x) = v_0 \quad \text{at} \quad x = 1,$$

then we can solve equation (4a), to get

$$u_1(x) = C_0\left(1 + 2ax^2 + \frac{8a^2 + b^2 a}{12}x^4 + \cdots\right),$$

where $\quad C_0 = 1\bigg/\left(1 + 2a + \frac{8a^2 + b^2 a}{12} + \cdots\right)$

$$a = \beta q_0 \ , \quad q_0 = \frac{v_0^2}{S^2} \ , \quad b = 2\sqrt{\beta\gamma} \ , \quad S = \left[e^{\sqrt{\beta\gamma}} + e^{-\sqrt{\beta\gamma}}\right],$$

and thus from boundary condition (4d) we get:

$$u_1(x) = 1 \quad at \quad x = 1$$

5. Disturbance case:

When the wave amplitudes are constants, stability analysis has been recently studied by numerous authors [6] and it is of great interest because of the growing industrial importance.

Assume that u_2 and v_2 has the following form [8]:

$$\begin{bmatrix} u_2(x,t) \\ v_2(x,t) \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} e^{ik(x-ct)} \qquad \ldots (7)$$

where $(c = c_1 + ic_2)$, is an eigenvalue represent the speed of the wave, y_1 and y_2 are constants and k is the wave number. The problem is stable if the linearized equation corresponds to eigenvalue c with negative part $(c_2 < 0)$ for presented configurations [5].

Now, if we substitute (7) in the equations (5a), we shall get:

$$-ikcy_1 e^{ik(x-ct)} = -k^2 y_1 e^{ik(x-ct)} - 2\beta u_1 v_1 y_2 e^{ik(x-ct)} - \beta v_1^2 y_1 e^{ik(x-ct)}$$

i.e. multiply both sides by $e^{-ik(x-ct)}$ we get:

$$-ikcy_1 = -k^2 y_1 - 2\beta u_1 v_1 y_2 - \beta v_1^2 y_1$$

$$-ik(c_1 + ic_2)y_1 = -k^2 y_1 - 2\beta u_1 v_1 y_2 - \beta v_1^2 y_1$$

$$-ikc_1 y_1 = 0 \quad \Rightarrow \quad c_1 = 0$$

$$kc_2 y_1 = -k^2 y_1 - 2\beta u_1 v_1 y_2 - \beta v_1^2 y_1$$

$$c_2 = \frac{-(k^2 + 2\beta u_1 v_1 \frac{y_2}{y_1} + \beta v_1^2)}{k}$$

where y_1, y_2, k, β are positive real number.

So if $(k^2 + 2\beta u_1 v_1 \frac{y_2}{y_1} + \beta v_1^2) < 0$ and u_1 and v_1 have different signs then we have $c_2 > 0$ and thus the system is unstable.

Also if $(k^2 + 2\beta u_1 v_1 \frac{y_2}{y_1} + \beta v_1^2) > 0$ and u_1 and v_1 have the same signs then we have $c_2 < 0$ which implies that the system is stable.

If $c_2 = 0$ then we can get the neutral curve.

Now if we substitute (7) in equation (5b) we shall get

$$-ikcy_2 e^{ik(x-ct)} = -k^2 y_2 e^{ik(x-ct)} + 2\beta u_1 v_1 y_2 e^{ik(x-ct)} + \beta v_1^2 y_1 e^{ik(x-ct)} - \beta \gamma y_2 e^{ik(x-ct)}$$

also if we multiply both sides by $e^{-ik(x-ct)}$ we get

$$-ik(c_1 + ic_2)y_2 = -k^2 y_2 + 2\beta u_1 v_1 y_2 + \beta v_1^2 y_1 - \beta \gamma y_2$$

$$-ikc_1 y_2 = 0 \quad \Rightarrow \quad c_1 = 0$$

$$kc_2 y_2 = -k^2 y_2 + 2\beta u_1 v_1 y_2 + \beta v_1^2 y_1 - \beta \gamma \ y_2$$

$$c_2 = \frac{-k^2 + 2\beta u_1 v_1 + \beta v_1^2 \frac{y_2}{y_1} - \beta \gamma}{k}$$

thus according to the above discussion we have three cases:

if $c_2 > 0$ then the system is unstable .

if $c_2 < 0$ then the system is stable.

if $c_2 = 0$ then get the neutral curve.

Then the neutral stability curve shows that in figure (1) when $c_2 < 0$ then $k = \sqrt{2\beta u_1 v_1 + \beta v_1^2 \dfrac{y_2}{y_1} - \beta\gamma}$ for the steady state solution v_1 is defined in eq.(6) with $v_0 = 0.1$ and $u_1 = 1$ when $\beta = 30$, $\gamma = 0.05$, y1=y2=1and $0 \le x \le 1$.
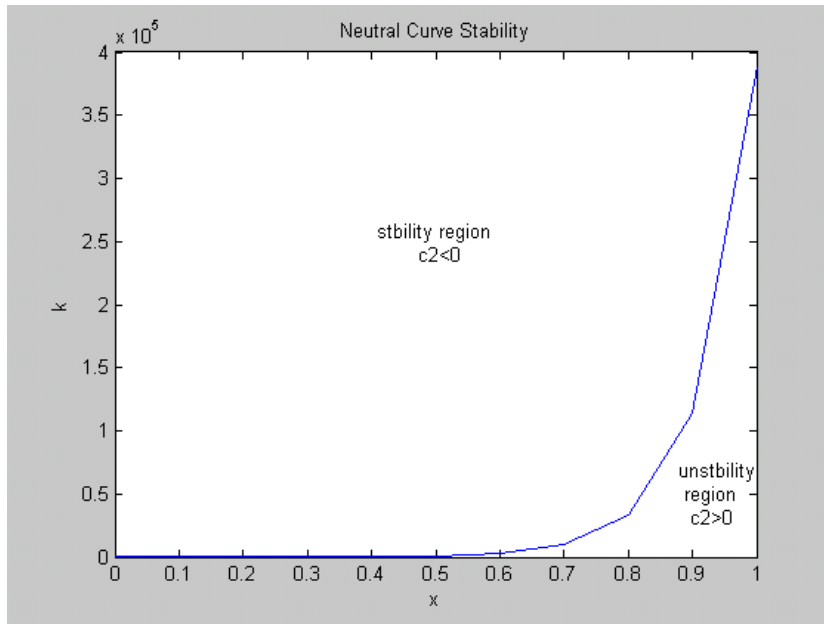

Figure (1) the neutral stability curve

6. CONCLUSION

From stability analysis of steady state solutions we conclude that the system is stable when $c_2 < 0$.

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