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The Strongly Tri-nil Clean Rings

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Abstract

This study explores the structure and properties of strongly Tri-nil clean rings. A ring is defined as strongly Tri-nil clean if every member in a ring can be expressed as the sum of a tripotent element and a nilpotent member, where these components commute. We provide the right singular ideal of a ring which is a nil ideal. We examine a ring with each elementσin R,$\sigma^2$is zhou. we also found in these rings the $char(R) = 48$, and every unit of order 4, Finally, we provide if R is a Tri-nil clean ring with $3 \in N(R)$ if and only if every member of R is a sum of three tripotent and nilpotent that commute.

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1. Introduction

Every ring in this study is associated with identity.

The Jacobson radical, the set of units of R, idempotents, tripotents, and nilpotents are represented by the symbols $J(R), U(R), Id(R), Tr(R)$, and $N(R)$, respectively.

We also identify Z_n as a ring of integers modulo n. A ring R is referred to as a stronglyTri-nil clean ring if each member can be expressed as the sum of a tripotent and a nilpotent that commute[1], [2], [3]The sum of a nilpotent and two tripotents that commute with one another is known as a Zhou ring [4].

When each member of R is the sum of a unit an idempotent, the ring is said to be a clean. As defined by W. K. Nicholson in 1997 [5]. Nicholson [6] defined the strongly clean ring of the unit and the idempotent commute later in 1999. Many authors worked in this kind of ring, for example [6], [7]

Disel[8] presented the idea of a nil-clean ring, A ring in which each member is the sum of an idempotent and a nilpotent. T. Kozan first proposed the idea of a strongly nil clean ring in 2016 [9].

2. Preliminaries

We provide well-known findings and definitions in this section that might be required in the subsequent.

Definition 2.1 [10] **:**

If $\rho = \rho^3$, then ρ is a tripotent element. If all of the elements of a ring R are tripotents, then the ring R is said to be tripotent.

Definition 2.2 [11] **:**

If there is a positive integer δ such that $\sigma^\delta = 0$, then a member σ of a ring R is referred to as nilpotent.

Lemma 2.3 [12] **:**

$(1 \mp n)$ and $u + n$ is a unit when n is a nilpotent, u is a unit, and $un = nu$.

Lemma 2.4 :

If ρ is a tripotent element ,then :

1. $\rho^2$and $1 - \rho^2$are idempotents.
2. $\rho^2 + \rho - 1$ is a unit of order 2.

Definition 2.5 [13] **:**

If $r(\sigma)$ is the right annihilator of σ, the right singular ideal of R is represented.

by $Y(R) = \{\sigma \in R : r(\sigma)$ is essential right ideal$\}$.

Definition 2.6 [10] **:**

A ring R is called a Zhou**,** if for every $\sigma \in R, \sigma = \rho_1 + \rho_2 + n$,where $\rho_1 , \rho_2 \in T(R), n \in N(R)$, that commute, with one another.

Example 2.6.1 :

In Z_{20} ring, we note that :

$N(R) = \{0,10\}$ and $Tr(Z_{20}) = \{0,1,4,5,9,11,15,16,19\}$,

clearly Z_{20} is a **Zhou**

Theorem 2.7 [14], [15] **:**

The following issues are equivalent for any ring R:

(1) R is strongly 2nil-clean.

(2) $\sigma^3 - \sigma \in N(R)$ for each $\sigma \in R$;

(3) each member in R is the sum of a commuting tripotent and a nilpotent.

Theorem 2.8 [16] **:**

Let R be a ring, the following are equivalent:

1. Zhou nil-clean is R.
2. $R/J(R)$ has the identity $\sigma^5 = \sigma$, and $J(R)$ is nil.
3. For all $\sigma \in R, \sigma^5 - \sigma$is nilpotent.

3. The strongly Tri-nil clean rings

Definition 3.1 [3] **:**

A tri-nil clean ring, or TNC for short, is a ring R. if $\sigma = \rho + n$, where $\rho^3 = \rho$, and $n \in N(R)$, for each $\sigma \in R$.Is referred to as a strongly TNC ring if $\rho n = n\rho$, or STNC for short.

Example 3.1.1 :

Take the ring Z_{12}.Then $N(Z_{12}) = \{0,6\}$ and $Tr(Z_{12}) = \{0,1,3,4,5,7,8,9,11\}$. Clearly the ring Z_{12} is a STNC ring.

Proposition 3.2 :

Every a STNC ring is a strongly Clean ring.

Proof :

Let $\sigma = \rho + n$, where $\rho \in Tr(R)$, $n \in N(R)$ and $\rho n = n\rho$, then σ may be written as $\sigma = 1 - \rho^2 + \rho^2 + \rho - 1 + n$, Note that $\rho^2 + \rho - 1$ is a unit, say u and $1 - \rho^2$ is idempotent, by Lemma2.5.2 say λ , $\sigma = \lambda + u$, Observe that $u\lambda = \lambda u$, therefore R is a strongly clean ring.

Proposition 3.3 :

A homomorphic image of a STNC ring is a STNC ring.

Proof :

Let F: $R \to R'$ be a homomorphism from R to R', for any $h \in R'$ existing $\sigma \in R$, so that $h = F(\sigma)$. Since $\sigma \in R$, $\sigma = \rho + n$, where $\rho^3 = \rho$, $n \in N(R)$, and $\rho n = n\rho$.

So , $h = F(\sigma) = (\rho + n) = F(\rho) + F(n)$,

Know we prove $F(\rho)$ is trepotent $F(\rho)^3 = F(\rho^3) = F(\rho)$ then $F(\rho) \in T(R')$, and $F(n) \in N(R')$, Thus $F(R)$ is a STNC ring is a homomorphic.

Proposition 3.4 :

If σ is a STNC element, then σ^2 is a strongly nil-clean.

Proof :

Let $\sigma \in R, \sigma = \rho + n$, where $\rho \in Tr(R)$, $n \in N(R)$, and $\rho n = n\rho, \sigma^2 = (\rho + n)^2 = \rho^2 + 2\rho n + n^2$

Since $\rho^2 \in Id(R)$,and $2\rho n + n^2 \in N(R)$. Thus σ^2 is a strongly nil-clean ring.

Proposition 3.5 :

If σ^2 is a STNC element, then σ and $-\sigma$ are strongly clean.

Proof :

Let $\sigma \in R, \sigma^2 = \rho + n$, where $\rho \in Tr(R)$, $n \in N(R)$, and $\rho n = n\rho$, we may write as :

$\sigma^2 = 1-\rho^2+\rho^2+\rho-1+n$, we notice that $u = \rho^2+\rho-1 \in U(R)$ and $\lambda = 1-\rho^2 \in Id(R)$ by Lemma 2.5.2 , we have $\sigma^2-\lambda = u$, Then $(\sigma-\lambda)(\sigma+\lambda) = u$,

$(\sigma-\lambda)(\sigma+\lambda)\ u^{-1} = 1$, then σ and $-\sigma$ are strongly clean ring.

Theorem 3.6 :

Let R be a STNC ring, then Y(R) is a nil ideal

Proof :

Let $\sigma \in Y(R)$, $\sigma = \rho+n$, such that $\rho \in Tr(R)$, $n \in N(R)$, and $\rho n = n\rho$.

since $\sigma \in Y(R)$, then $r(\sigma)$ is essential right ideal. Consider $r(\sigma) \cap \rho R$.

Take $x \in r(\sigma) \cap \rho R$, we have $x \in r(\sigma)$and $x \in \rho R$,

we have $\sigma x = 0$ and $x = \rho r$. So $\sigma\rho r = 0$ then$(\rho+n)\rho r = 0$,

$\rho^2 r + n\rho r = 0$,hence $\rho^2 r + n\rho^3 r = 0$, then $\rho^2 r(1+n\rho) = 0$, $\rho^2 r u = 0$, gives $\rho^3 r = 0$, thus $\rho r = x = 0$

As $r(\sigma)$ is a nontrivial essential ideal, then $\rho R = 0$, gives $\rho = 0$.

Theorem 3.7 :

If R is a ring with every $\sigma \in R$, σ^2 is a STNC ring, then R is zhou ring.

Proof :

Let $\sigma^2 \in R$, $\sigma^2 = \rho+n$, where $\rho \in Tr(R)$, $n \in N(R)$, and $\rho n = n\rho$, by Theorem2.9, then we have

$(\sigma^2)^3-\sigma^2 \in N(R)$, gives $\sigma^6-\sigma^2 \in N(R)$, so $\sigma(\sigma^5-\sigma) \in N(R)$

Thus $(\sigma^4-1)\sigma(\sigma^5-\sigma) \in N(R)$, then$(\sigma^5-\sigma)^2 \in N(R)$,

we get $\sigma^5-\sigma \in N(R)$, therefore R is zhou ring.

Theorem 3.8 :

If R is a STNC ring with $n^2+2n = 0$, for every $n\epsilon N(R)$ then $char(R) = 48$, and every unit of order 4.

Proof :

Let $\sigma \in R$, $\sigma = \rho+n$, where $\rho \in Tr(R)$, $n \in N(R)$, and $\rho n = n\rho$,

For any $u \in U(R)$,then $u = \rho+n$, gives $u-n = \rho$.

By Lemma2.5.1 $u-n \in U(R)$, this implies $\rho^2 = 1$.

On the other hand $u^2 = \rho^2+2\rho n+n^2$,so $u^2 = \rho^2+n'$,where $n' = 2n\rho+n^2 \in N(R)$,so $u^2 = 1+n'$,

Now $u^4 = (1+n')^2 = 1+2n'+n'^2$(since $2n'+n'^2 = 0$),by assumption, then $U^4 = 1$.

Furthermore, since $6 \in N(R)$by Proposition3.1,and since $n^2+2n = 0$,for every $n \in N(R)$.

Then, $6^2+2(6) = 0, 36+12 = 0$,then $48 = 0$.

Example 3.8.1 :

Consider Z_{48} , where

$N(Z_{48}) = \{0,6,12,18,24,30,36,42\}$,and

$U(Z_{48}) = \{1,5,7,11,13,17,19,23,25,29,31,35,41,43,47\}$

Now that $U^4 = 1$,for every $u \in U(R)$

Theorem 3.9 :

Let R be a STNC ring, then for every $\sigma \in R$, there is at least $b \in R$ such that $\sigma b \in Id(R)$, or $\sigma b \in Tr(R)$.

Proof :

Let $\sigma \in R$, then $\sigma = \rho+n$, where $\rho \in Tr(R)$ and $n \in N(R)$, and $\rho n = n\rho$.

Since n is nilpotent, then ,$n^{\delta} = 0$, for some positive integer ς. So

$b = \rho^{\delta-1}-\rho^{\delta-2}n+\rho^{\delta-3}n^2-\rho^{\delta-4}n^3+\rho^{\delta-5}n^4-\cdots+(-1)^{\delta+1}n^{\delta-1}$

$\sigma b = (\rho+n)(\rho^{\delta-1}-\rho^{\delta-2}n+\rho^{\delta-3}n^2-\rho^{\delta-4}n^3+\rho^{\delta-5}n^4-\cdots+(-1)^{\delta+1}n^{\delta-1})$

$\sigma b = (\rho^{\delta}-\rho^{\delta-1}n+\rho^{\delta-2}n^2-\rho^{\delta-3}n^3+\rho^{\delta-4}n^4-\cdots+(-1)^{\delta+1}\rho n^{\delta-1}+\rho^{\delta-1}n-\rho^{\delta-2}n^2+\rho^{\delta-3}n^3-\rho^{\delta-4}n^4+\cdots+(-1)^{\delta+1}n^{\delta}$,

$\sigma b = \rho^{\delta}$

If δ is even , then $\sigma b = \rho^2$, $\rho^2 \in Id(R)$, or δ is odd, then $\sigma b = \rho$, $\rho \in Tr(R)$.

Theorem 3.10 :

For a ring with $3 \in N(R)$,then R is a STNC ring if and only if every member of R is a sum of 3 tripotents and nilpotent that commute.

Proof :

Let $\sigma \in R$, $\sigma = \rho+n$, where $\rho \in Tr(R)$, $n \in N(R)$, and $\rho n = n\rho$

$\sigma = 0+0+\rho+n$. As a result, σ is the sum of 3 tripotents that commute.

Conversely: Let $\sigma = \rho_1 + \rho_2 + \rho_3 + n$, where $\rho_1, \rho_2, \rho_3 \in Tr(R)$ and $n \in N(R)$, that commute with each other

Now,$\sigma^3 = (\rho_1 + \rho_2 + \rho_3 + n)^3 = (\rho_1 + \rho_2 + \rho_3)^3 + 3(\rho_1 + \rho_2 + \rho_3)^3 n + 3(\rho_1 + \rho_2 + \rho_3)^3 n^2 + n^3, = \rho_1 + \rho_2 + \rho_3 + n'$.

So,$(\rho_1 + \rho_2 + \rho_3)^3 = (\rho_1 + \rho_2)^3 + 3(\rho_1 + \rho_2)^2 \rho_3 + 3(\rho_1 + \rho_2) \rho_3{}^2 + \rho_3{}^3$.

since $3 \in N(R)$, then $3(\rho_1 + \rho_2)^2 \rho_3 + 3(\rho_1 + \rho_2) \rho_3{}^2 \in N(R)$, say n.

And $(\rho_1 + \rho_2)^3 = \rho_1^3 + 3\rho_1^2\rho_2 + 3\rho_1\rho_2^2 + \rho_2^3$

Since $3 \in N(R)$then $n' = 3\rho_1^2\rho_2 + 3\rho_1\rho_2^2 \in N(R)$,and $n + n' = n'' \in N(R)$

Therefore $\sigma^3 = \rho_1^3 + \rho_2^3 + \rho_3^3 + n''$,$\sigma^3 = \rho_1 + \rho_2 + \rho_3 + n''$,

Then $\sigma^3 - \sigma = n'' - n$,so $\sigma^3 - \sigma \in N(R)$ by Theorem 2.8, σ is a STNC ring.

Conclusion

We demonstrate that the Jacobson radical and the right singular ideal over a strongly Tri-nil clean rings are nil ideals in this work, which gives a new property of a strongly Tri-nil clean ring. Additionally, provide the relationships between rings that are strongly Tri-nil clean and rings that are related. In addition, we study a ring with each of its two members σ, b in R,$\sigma.b = \rho$, $\sigma b \in Id(R)$, or $\sigma b \in Tr(R)$. and we introduce and study a particular class of strongly Tri-nil clean rings. Lastly, we demonstrate that a ring that contains all of the elements σ in R,σ^2 is a zhou ring that is strongly Tri-nil clean.

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Conflict of interest

None.

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