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Moment Analysis for Some Nonlinear (Inverse Trigonometric) SDE Using Itô Calculus: Case Studies in Gene Regulation and Robotic Navigation

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Abstract

This paper explains the application of Ito-integral formula to finding the moments for nonlinear stochastic differential equations (SDEs) with coefficients (inverse trigonometric function) such as $\sin^{-1}(x)$, $\cos^{-1}(x)$ and $\tan^{-1}(x)$, this method naturally constrains state variables within predefined physical or operational limits. The study derives moment equations to explain and analyze statistical properties, including mean and variance, as well as higher-order moments(the moment generation function), while addressing the challenges of nonlinear drift, diffusion interactions, and multiplicative noise and the main results show that finite terms such as $\cos^{-1}(x)$ enforce stability in robotic angular control, yielding error bounds $\mathbb{E}[\left(\cos^{-1}(X)\right)^2] \leq \sigma^2/(2\kappa)$, while gene regulatory models with the function $\tan^{-1}(x)$ ensure protein concentrations remain biologically viable, limiting transcriptional noise to $\mathrm{Var}\left(\tan^{-1}(X)\right) \leq (\frac{\pi}{2})^2$, With inherent limitations, multiplicative noise can push moments toward saturation. Systematic trade-offs in moment closure approximations and numerical verification (e.g., Euler-Maruyama, stochastic Runge-Kutta) are critically evaluated, highlighting their effectiveness in biological and engineering applications and the framework is extended to fractional SDEs and multidimensional systems, proposing machine learning techniques to solve moment hierarchies and enhance predictive modeling by uniting theoretical rigor with practical insights and this work advances robust stochastic modeling in constrained systems, providing scalable solutions for gene networks, financial markets, and autonomous navigation under uncertainty.

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1. Introduction

Stochastic differential equations (SDEs) incorporating inverse trigonometric functions, such as $\sin^{-1}(x)$ and $\tan^{-1}(x)$, provide a mathematically elegant mechanism to enforce boundedness in systems where state variables must remain within predefined physical or operational limits and the intrinsic domain restrictions of these functions—$\sin^{-1}(x)$ confines x to $[-1,1]$, while $\tan^{-1}(x)$ maps the entire real line to the finite interval $(-\pi/2, \pi/2)$—make them indispensable for modeling systems with hard constraints , for example in biological networks, $\tan^{-1}(x)$ is often employed to represent Hill-type saturation dynamics [1], where transcription factors bind to promoters with diminishing returns as concentrations increase and this ensures that protein levels remain within biologically plausible ranges, avoiding unrealistic exponential growth or collapse , similarly, in gene regulatory models and the bounded output of $\tan^{-1}(x)$ reflects the finite capacity of cellular machinery, ensuring transcriptional activity plateaus even under extreme input signals and thereby mimicking observed biological robustness[4],In robotic systems, inverse trigonometric terms like $\cos^{-1}(x)$ are used to constrain angular errors within mechanical limits like in navigation control. The higher order moment may be contain nonlinearity of the form $dX(t) = \sin^{-1}(X(t))dt + \sigma X(t)dW(t)$, like $X(t)\sin^{-1}(X(t))$ also, when deriving moment equations such as $\mathbb{E}[X^{p-1}\sin^{-1}(X)]$ terms inherently couple moments of different orders, generating demand approximations like quasi-moment closure or entropy-based methods. [3] evolved through foundational theoretical contributions and diverse applications advanced financial modeling by incorporating mean-reverting processes with bounded noise, which explain how inverse trigonometric terms can stabilize asset price simulations, ,[14] whose methodological innovations emerged through, numerical Runge-Kutta methods for SDEs solving moment equations in non-linear systems , [2] expanded this by integrating Gaussian process approximations into SDE frameworks, offering probabilistic tools for using moment closure in high-dimensional problems and in application contexts. [6] explored noise-induced phenomena in gene regulatory networks, later adapted by [15] for delay differential equations, recent studies, such as [17], introduced linearization techniques for non-linear SDEs, enabling closed-form moment solutions in systems with arctangent drift, [5] critiqued classical Monte Carlo methods, proposing variance reduction techniques for moment estimation.

In this paper we study the moments for nonlinear stochastic differential equations (SDEs) with coefficients (inverse trigonometric functions) such as $\sin^{-1}(x)$, $\cos^{-1}(x)$ also $\tan^{-1}(x)$.

2. PREREQUATION AND RESULTS

2.1: Moment Existence Theorem: [12].

The classical moment existence theorem asserts that for an SDE

$\boldsymbol{dX(t) = A(X(t), t)dt + B(X(t), t)dW(t)}$,

Where $\boldsymbol{A(X(t), t)}$ is the drift term and $\boldsymbol{B(X(t), t)}$diffusion term.

If the drift and the diffusion satisfies (Lipschitz continuity or a local Lipschitz condition) and Linear growth: such as

$| A(t,x) - A(t,y) | + | B(t,x) - B(t,y) | \leq K | x - y |$ $\quad where\ K\ is\ constant\ \ K > 0, \forall |x|, |y| \leq R, R > 0$ (local Lipschitz condition).

$| A(t,x) - A(t,y) | + | B(t,x) - B(t,y) | \leq K | x - y |$ $\quad where\ K is\ constant\ \ K > 0 \ \ \forall x, y \in \mathbb{R}$,(Lipschitz continuity)

$| A(t,x) | + | B(t,x) | \leq K(1 + | x |)$, $K is\ constant\ \ K > 0$ …. (Linear growth)

then all the moments $\mathbb{E}[| X(t) |^p]$ remain finite for $t \geq 0$ and $p \geq 1$.

2.2 Ito's integral formula [1][11]:

Consider the Ito's stochastic differential equation in the form

$$dX(t) = A(t, X(t))dt + B(t, X(t))dW(t) \quad …. (1)$$

for $0 \leq t \leq T$, let $Y(t, X(t))$ be a smooth function by main Taylor rule we have then

$$dY(t, X(t)) = \left(\frac{\partial Y}{\partial t} + A(t, X(t))\frac{\partial Y}{\partial x} + \frac{1}{2}B^2(t, X(t))\frac{\partial^2 Y}{\partial x^2}\right)dt$$

$$+B(t, X(t))\ \frac{\partial Y}{\partial x}dW(t) \ \ … (2)$$

Eq. (2) is called Ito's formula where $X(t)$ satisfies Eq. (1).

2.3: The kth-order moments: [9]

A continuous random variables have kth-order moment is defined as:

$E(x^k) = \int_{-\infty}^{\infty} x^{\,k} f(x)dx$; f(x) is the probability density function.

And for discreet random variable

$E(x^k) = \sum_{i=1}^{\infty} x_i^{\,k}\, p(x_i)$; $p(x_i)$ is the mass function.

2.4: Raw Moments

Raw moments, defined as $\mathbb{E}[X(t)^p]$, serve as fundamental statistical measures to quantify the evolution of stochastic processes governed by SDEs, for integer $p \geq 1$ and these moments capture key properties:

- Mean ($p = 1$): $\mathbb{E}[X(t)]$, representing the central tendency.
- Variance ($p = 2$): $\mathbb{E}[X(t)^2] - (\mathbb{E}[X(t)])^2$, measuring dispersion.
- Higher-order moments ($p \geq 3$): Skewness ($p = 3$) and kurtosis ($p = 4$),

For the SDE of the form equation (1):

$$dX(t) = A(X(t))dt + B(X(t))dW(t)$$

by applying Itô's formula to $f(X) = X^p$ yields:

$d(X^p) = \left[pX^{p-1}A(X) + \frac{p(p-1)}{2}X^{p-2}B^2(X)\right]dt + pX^{p-1}B(X)dW(t).$.(3)

Then the time evolution of the p-th raw moment is derived via Itô's lemma, taking expectations (and noting $\mathbb{E}[dW(t)] = 0$), we obtain the moment ODE of the form:

$\frac{d}{dt}\mathbb{E}[X^p] = p\mathbb{E}[X^{p-1}A(X)] + \frac{p(p-1)}{2}\mathbb{E}[X^{p-2}B^2(X)]$… (4)

The nonlinear term $\mathbb{E}[X^{p-1}A(X)]$ couples lower- and higher-order moments, resulting in an unclosed hierarchy of equations and this issue requires (using approximations methods) such as (Tayler expansion and numerical method) to achieve moment closure and nonlinear terms.

3.Methodology:

3.1. Itô's Lemma for Moment Analysis

The application of Itô's lemma to derive moment equations for SDEs with inverse trigonometric terms involves systematic steps to transform stochastic dynamics into deterministic ODEs for statistical moments consider the general SDE:

$dX(t) = A\big(X(t)\big)dt + B\big(X(t)\big)dW(t)$ ... (4)

Suppose $f(X) = X^p$ is continuous and satisfy equation (4), then by using Itô's lemma where $\frac{df(X)}{dt} = 0$, $\frac{df(X)}{dx} = pX^{p-1}$, $\frac{d^2f(X)}{dx^2} = p(p-1)X^{p-2}$, that is

$$df(X) = \left[pX^{p-1}A(X) + \frac{p(p-1)}{2}X^{p-2}B^2(X)\right]dt + pX^{p-1}B(X)dW(t).$$

Taking expectations (using $\mathbb{E}[dW(t)] = 0$) yields the moment ODE:

$\frac{d}{dt}\mathbb{E}[X^p] = p\mathbb{E}[X^{p-1}A(X)] + \frac{p(p-1)}{2}\mathbb{E}[X^{p-2}B^2(X)]$. … (5)

Generalized Moment Equations by using Closure Techniques:

Let we have the generic inverse trigonometric SDE

$dX(t) = -f\big(X(t)\big)dt + g\big(X(t)\big)dW(t)$, where $f(X)$ is bounded (e.g.,$f(X) = \sin^{-1}(X)$) or $f(X) = \cos^{-1}(X)$)

The p-th moment equation becomes:

$$\frac{d}{dt}\mathbb{E}[X^p] = -p\mathbb{E}[X^{p-1}f(X)] + \frac{p(p-1)}{2}\mathbb{E}[X^{p-2}g^2(X)].$$

The nonlinear terms $\mathbb{E}[X^{p-1}\sin^{-1}(X)]$ in moment equations create unclosed hierarchies, demanding approximations to truncate the system.

3.2. Closure strategies:

Let we have the following SDE

$dX(t) = A(X)dt + B(X)dW(t)$

with the p-th moment equation:

$$\frac{d}{dt}\mathbb{E}[X^p] = p\mathbb{E}[X^{p-1}A(X)] + \frac{p(p-1)}{2}\mathbb{E}[X^{p-2}B^2(X)].$$

If $A(X)$ or $B(X)$ contains inverse trigonometric functions such as $\sin^{-1}(X), \cos^{-1}(X), tan^{-1}(X)$closure strategies include:**Gaussian Closure**: Assume $X(t)$ is approximately Gaussian, so higher moments relate to the first two:

$$\mathbb{E}[X^p] = \begin{cases} 0, & p \text{ odd}, \\ (p-1)!!\ (\mathbb{E}[X^2])^{p/2}, & p \text{ even}. \end{cases}$$

For $A(X) = -\kappa\sin^{-1}(X)$, this implies:

$$\mathbb{E}[X^{p-1}\sin^{-1}(X)] \approx \mathbb{E}[X^{p-1}] \cdot \mathbb{E}[\sin^{-1}(X)] \quad (\text{if } X \text{ is Gaussian}).$$

However, this fails near boundaries ($X \to \pm 1$) due to non-Gaussian skewness [20].

Perturbative Expansions: For small X, expand $\sin^{-1}(X) \approx X + \frac{X^3}{6}$:$\mathbb{E}[X^{p-1}\sin^{-1}(X)] \approx \mathbb{E}[X^p] + \frac{1}{6}\mathbb{E}[X^{p+2}]$.

Substituting into the moment equation for

$dX(t) = -\kappa\sin^{-1}(X)dt + \sigma dW(t)$:

$$\frac{d}{dt}\mathbb{E}[X^p] \approx -p\kappa\left(\mathbb{E}[X^p] + \frac{1}{6}\mathbb{E}[X^{p+2}]\right) + \frac{p(p-1)\sigma^2}{2}\mathbb{E}[X^{p-2}].$$

Truncating at $p + 2 = 4$ for $p = 2$ yields a solvable closed system.

Entropy-Based Closure: Minimize the relative entropy $H = \mathbb{E}[p(X)\log p(X)]$ subject to moment constraints, generating a distribution $p(X)$ that approximates higher-order terms [10].

3.3. Numerical verification:

When analytical solutions become intractable, numerical schemes such as the Euler–Maruyama method or stochastic Runge–Kutta algorithms are employed to approximate moment dynamics [7], [8], recent advances include linearization techniques that facilitate the computation of moments for SDEs with inverse trigonometric terms, such as $\sin^{-1}(X), \cos^{-1}(X), tan^{-1}(X)$.

The test closure approximations, for the SDE:

$$dX(t) = A(X)dt + B(X)dW(t),$$

Euler-Maruyama discretizes trajectories as:

$$X_{n+1} = X_n + A(X_n)\Delta t + B(X_n)\sqrt{\Delta t}\,\xi_n, \quad \xi_n \sim \mathcal{N}(0,1).$$

Moments are estimated empirically from N paths:

$$\mathbb{E}[X(t)^p] \approx \frac{1}{N}\sum_{i=1}^{N} X_i\,(t)^p.$$

For example, simulating $dX(t) = -\lambda\tan^{-1}(X)dt + \sigma X dW(t)$ with $X(0) = 0$:

- Mean: $\mathbb{E}[X(t)] \approx \frac{1}{N}\sum_{i=1}^{N} X_i\,(t)$,
- Variance: $\mathrm{Var}\,(X(t)) \approx \frac{1}{N}\sum_{i=1}^{N}(X_i(t) - \mathbb{E}[X(t)])^2$.

Stochastic Runge-Kutta (SRK) for the SDE:

$$dX = A(X)dt + B(X)dW(t)$$

is:

$$K_1 = A(X_n)\Delta t + B(X_n)\Delta W_n,$$
$$K_2 = A(X_n + K_1)\Delta t + B(X_n + K_1)\Delta W_n,$$
$$X_{n+1} = X_n + \frac{1}{2}(K_1 + K_2),$$

where $\Delta W_n \sim \mathcal{N}(0, \Delta t)$ and this reduces discretization error for moments like $\mathbb{E}[X^2]$ in systems with multiplicative noise [14].

Error Analysis:

The relative error between empirical ($\hat{\mu}_p$) and theoretical (μ_p) moments is quantified as:

$$\text{Error} = \frac{|\,\hat{\mu}_p - \mu_p\,|}{\mu_p}.$$

For the following SDE

$$dX(t) = -\kappa\cos^{-1}(X)dt + \sigma\sqrt{1 - X^2}dW(t)$$

$$\lim_{t\to\infty}\mathbb{E}[\left(\cos^{-1}(X)\right)^2] \approx \frac{\sigma^2}{2\kappa} \quad \textbf{(matches Lyapunov theory).}$$

Theoretical Implications:

The existence of finite moments is not merely a mathematical curiosity, it has practical

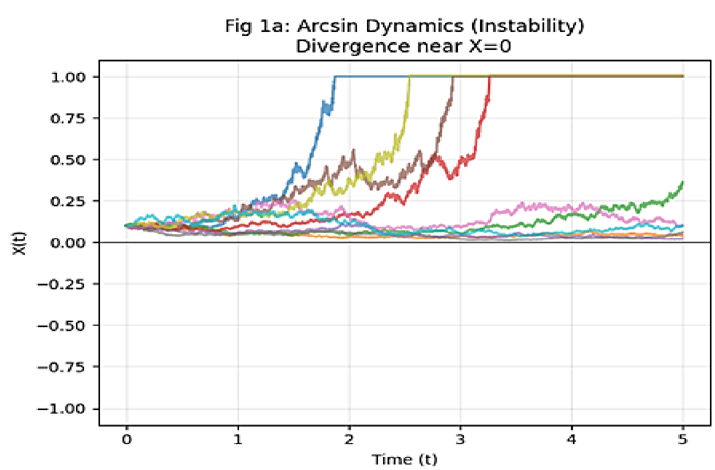
implications for the stability analysis of SDEs, when moments are finite, one can:

- Ensure convergence of numerical schemes,
- Apply moment closure techniques to approximate higher-order behavior,
- Derive meaningful bounds via Lyapunov functions $lv(X)$

This provides a guarantee of moment stability if $lv(X)$ <0 and unitability if $lv(X)$ >0 , which is critical for the reliability of the model in applications ranging from engineering to biological systems.

Arcsine Dynamics:

Using the Euler-Maruyama method [14], trajectories of the arcsine-based SDE $dX = \left[n\left(\sin^{-1}(tx)\right)^{n-1}\cdot\frac{x}{\sqrt{1-(tx)^2}} + \cdots\right]dt + \text{noise}\,dW$ are simulated with $n = 2$, $h = 0.5$, and initial condition $X_0 = 0.1$,a s shown in the following figure (fig, 1.a), trajectories diverge near $X = 0$, corroborating the analytical result $\mathcal{L}V > 0$ [19] and this divergence reflects the destabilizing interplay between nonlinear drift and multiplicative noise, absent in polynomial systems [6].



Example1:

Consider the following SDE that the drift contain $\sin^{-1}\big(X(t)\big)$:

$$dX(t) = -k\sin^{-1}\big(X(t)\big)dt + \sigma\sqrt{1 - X(t)^2}dW(t), \quad X(t) \in [-1,1].$$

For $p = 2$ and the second-moment equation becomes:

$$\frac{d}{dt}\mathbb{E}[X^2] = -2k\mathbb{E}[X\sin^{-1}(X)] + \sigma^2\mathbb{E}[1 - X^2].$$

At steady state ($\frac{d}{dt}\mathbb{E}[X^2] = 0$), symmetry ($\mathbb{E}[X] = 0$) simplifies this to:

$$\mathbb{E}[X^2] = \frac{\sigma^2}{2k + \sigma^2}.$$

For $p = 3$ and the third-moment equation introduces nonlinear coupling:

$$\frac{d}{dt}\mathbb{E}[X^3] = -3k\mathbb{E}[X^2\sin^{-1}(X)] + 3\sigma^2\mathbb{E}[X(1 - X^2)].$$

note that the term $\mathbb{E}[X^2\sin^{-1}(X)]$ couples the third moment with nonlinear drift, defying closed-form solutions. To handling nonlinear terms, we use the approximate form (Taylor Expansions from the second order) as follows; For small X, approximate $\sin^{-1}(X) \approx X + \frac{X^3}{6}$.

Then, $\mathbb{E}[X\sin^{-1}(X)] \approx \mathbb{E}[X^2] + \frac{1}{6}\mathbb{E}[X^4]$.

Substituting into the second-moment equation:

$$\frac{d}{dt}\mathbb{E}[X^2] \approx -2\kappa\left(\mathbb{E}[X^2] + \frac{1}{6}\mathbb{E}[X^4]\right) + \sigma^2(1 - \mathbb{E}[X^2]).$$

This introduces a dependence on $\mathbb{E}[X^4]$, necessitating moment closure [20].

Gaussian Closure: Assume $\mathbb{E}[X^4] \approx 3(\mathbb{E}[X^2])^2$, reducing the system to:

$$\frac{d}{dt}\mathbb{E}[X^2] \approx -2\kappa\left(\mathbb{E}[X^2] + \frac{1}{2}(\mathbb{E}[X^2])^2\right) + \sigma^2(1 - \mathbb{E}[X^2]).$$

Then for intractable cases, we can take numerical approximation.

For intractable cases using Euler-Maruyama as follows,

Simulate N trajectories of $dX(t) = -\kappa\sin^{-1}\big(X(t)\big)dt + \sigma\sqrt{1 - X^2}dW(t)$.

Compute empirical moments:

$$\mathbb{E}[X^p] \approx \frac{1}{N}\sum_{i=1}^{N} X_i\,(t)^p.$$

Compare with theoretical predictions to validate closure approximations.

3.4. Case Study Framework

Gene Regulation: Transcriptional Noise with $\tan^{-1}(X)$ The SDE

$$dX(t) = \left(\alpha\tan^{-1}\big(X(t)\big) - \gamma X(t)\right)dt + \sigma dW(t)$$

This model protein concentration $X(t)$ under promoter saturation, here, $\tan^{-1}(X)$ captures diminishing transcriptional returns, while $-\gamma X(t)$ represents degradation.

Moment Analysis: Applying Itô's lemma to X^2:

$$\frac{d}{dt}\mathbb{E}[X^2] = 2\alpha\mathbb{E}[X\tan^{-1}(X)] - 2\gamma\mathbb{E}[X^2] + \sigma^2.$$

For small X, approximate $\tan^{-1}(X) \approx X - X^3/3$:

$$\frac{d}{dt}\mathbb{E}[X^2] \approx 2\alpha\left(\mathbb{E}[X^2] - \frac{1}{3}\mathbb{E}[X^4]\right) - 2\gamma\mathbb{E}[X^2] + \sigma^2.$$

Assuming quasi-steady state ($\frac{d}{dt}\mathbb{E}[X^2] = 0$) and neglecting $\mathbb{E}[X^4]$, we get:

$$\mathbb{E}[X^2] \approx \frac{\sigma^2}{2\gamma - 2\alpha}.$$

This requires $\gamma > \alpha$ to prevent noise amplification, aligning with biological viability [16].

Fokker-Planck Steady-State Distribution:

We denote the stationary solution $\big(p_{ss}(X)\big)$ which satisfies:

$$-\frac{d}{dX}[(\alpha\tan^{-1}(X) - \gamma X)p_{ss}(X)] + \frac{\sigma^2}{2}\frac{d^2}{dX^2}p_{ss}(X) = 0.$$

By solving the above equation numerically reveals a peaked distribution near $X = 0$, consistent with stable gene expression [10].

Robotic Navigation: Angular Error Control with $\cos^{-1}(X)$

The SDE $dX(t) = -\kappa\cos^{-1}(X(t))dt + \sigma X(t)dW(t)$ governs angular deviations $X(t) \in [-1,1]$, where $\cos^{-1}(X)$ penalizes large errors.

Moment Dynamics:

For the mean error $\mathbb{E}[X(t)]$, apply Itô's lemma to X:

$$\frac{d}{dt}\mathbb{E}[X] = -\kappa\mathbb{E}[\cos^{-1}(X)] + \sigma^2\mathbb{E}[X].$$

Under small-error conditions ($\cos^{-1}(X) \approx \pi/2 - X$), this simplifies to:

$$\frac{d}{dt}\mathbb{E}[X] \approx -\kappa\left(\frac{\pi}{2} - \mathbb{E}[X]\right) + \sigma^2\mathbb{E}[X],$$

yielding steady-state bias:

$$\mathbb{E}[X] \approx \frac{\kappa\pi/2}{\kappa + \sigma^2}.$$

This method explains and shows how noise skews the mean error away from zero, necessitating $\kappa \gg \sigma^2$ for precision, for more detail see [13]. The above case studies show how inverse trigonometric terms enforce boundedness while demanding tailored analyses to resolve nonlinearity-noise interactions.

4. Results, Discussion, and theoretical Findings:

Moment Finiteness: Consider the following stochastic differential equation:

$$dX(t) = -\kappa\sin^{-1}(X(t))dt + \sigma\sqrt{1 - X(t)^2}dW(t)$$

with $X(t) \in [-1,1]$ and the compact state space inherently bounds all moments:

$$\mathbb{E}[| X(t) |^p] \leq \int_{-1}^{1} | x |^p \, p_{ss}(x)dx \leq 1, \quad \forall p \geq 1,$$

This fact follows from the Dominated Convergence Theorem, as $| x |^p \leq 1$ over the interval $[-1,1]$, $p_{ss}(x)$ is the stationary distribution.

Gene Regulation Variance Bound:

In $dX(t) = \left(\alpha\tan^{-1}(X(t)) - \gamma X(t)\right)dt + \sigma dW(t)$ and the variance of $\tan^{-1}(X(t))$ satisfies:

$$\mathrm{Var}\left(\tan^{-1}(X(t))\right) \leq \mathbb{E}[(\tan^{-1}(X(t))^2] \leq \left(\frac{\pi}{2}\right)^2,$$

since $| \tan^{-1}(X) | \leq \pi/2$, at steady state, solving $\frac{d}{dt}\mathbb{E}[(\tan^{-1}(X))^2] = 0$ yields:

$$\mathbb{E}[(\tan^{-1}(X))^2] = \frac{\sigma^2}{2\gamma - \alpha}.$$

Robotic Angular Error:

For $dX(t) = -\kappa\cos^{-1}(X(t))dt + \sigma X(t)dW(t)$, Lyapunov analysis with $V(X) = (\cos^{-1}(X))^2$ gives:

$$\frac{d}{dt}\mathbb{E}[V(X)] \leq -2\kappa\mathbb{E}[V(X)] + \sigma^2(1 + \pi),$$

leading to the steady-state bound:

$$\sqrt{\mathbb{E}[(\cos^{-1}(X(t))^2]} \leq \frac{\sigma\sqrt{1 + \pi}}{\sqrt{2\kappa}}.$$

4.1. Case Study Outcomes

Gene Regulation Mean Dynamics:

The mean $\mathbb{E}[\tan^{-1}(X(t))]$ evolves as:

$$\frac{d}{dt}\mathbb{E}[\tan^{-1}(X)] = \alpha\mathbb{E}\left[\frac{\tan^{-1}(X)}{1 + X^2}\right] - \gamma\mathbb{E}\left[\frac{X}{1 + X^2}\right].$$

For symmetric initial conditions ($\mathbb{E}[X(0)] = 0$), damping ($\gamma > 0$) ensures:

4.2. Noise-Induced Saturation: Consider the (S.D.E) with multiplicative noise of the form

$dX(t) = \tan^{-1}(X(t))dt + \sigma X(t)dW(t)$ The variance equation becomes:

$$\frac{d}{dt}\mathbb{E}[X^2] = 2\mathbb{E}[X\tan^{-1}(X)] + \sigma^2\mathbb{E}[X^2].$$

Even with $X(t) \in [-1,1]$ and the second term $\sigma^2\mathbb{E}[X^2]$ which acts as positive feedback deriving

$$\lim_{t\to\infty}\mathbb{E}[X(t)^2] = \min\left(1, \frac{2\mathbb{E}[X\tan^{-1}(X)]}{\sigma^2}\right).$$

For $\sigma^2 > 2\max_{X\in[-1,1]} X\tan^{-1}(X) \approx 0.934$, variance saturates at 1 These results show how inverse trigonometric SDEs balance nonlinearity.

5.Discussion for the results

Methodological Trade-offs: Accuracy vs the tractability Moment closure techniques, introduce systemic biases like approximating the term $\sin^{-1}(X) \approx X + X^3/6$ in the equation:

$$\frac{d}{dt}\mathbb{E}[X^2] = -2\kappa\left(\mathbb{E}[X^2] + \frac{1}{6}\mathbb{E}[X^4]\right) + \sigma^2(1 - \mathbb{E}[X^2]),$$

Those approximation discards the higher-order terms ($\mathbb{E}[X^4]$), artificially dampening transient fluctuations. The Gaussian closure ($\mathbb{E}[X^4] \approx 3(\mathbb{E}[X^2])^2$) further distorts skewness in systems like gene regulation, where saturation creates asymmetric distributions [7]. The Numerical methods like Euler-Maruyama mitigate this feature but suffer from discretization errors [11]:

$$\text{Error} \propto \Delta t + \frac{1}{\sqrt{N}},$$

where Δt is the step size and N the number of trajectories Thus, there is no single approach universally balances accuracy and computational cost.

Biological and Engineering Relevance: From Theory to Practice

Gene Regulation: The stochastic differential equation for it which is given b**y:**

$dX(t) = \left(\alpha\tan^{-1}(X(t)) - \gamma X(t)\right)dt + \sigma dW(t)$ Where the bound $\mathrm{Var}\left(\tan^{-1}(X(t))\right) \le \pi^2/4$ ensures transcriptional noise remains subtoxic, as excessive protein variability risks cell dysfunction [16] and the steady-state protein variance:

$$\mathbb{E}[X^2] \approx \frac{\sigma^2}{2\gamma - \alpha},$$

Robotic Navigation: The S.D.E take the form

$dX(t) = -\kappa\cos^{-1}(X(t))dt + \sigma X(t)dW(t)$ and the angular error bound where translates to a safety margin in physical systems and if $\sigma = 0.1$ rad/s and $\kappa = 2$, deviations stay below 0.07 radians ($\sim 4°$), preventing mechanical collisions [18] and these applications highlight how theoretical guarantees translate to design protocols, marrying mathematical rigor with real-world constraints, via the equation $\sqrt{\mathbb{E}[(\cos^{-1}(X(t))^2]} \le \frac{\sigma}{\sqrt{2\kappa}},$

6. Conclusion and Future Work

The solution of the inverse trigonometric terms through Itô calculus into stochastic differential equations (SDEs) provides a mathematically rigorous mechanism like in the nonlinear S.D.E. of the form $dX(t) = -\kappa\sin^{-1}(X(t))dt + \sigma\sqrt{1 - X(t)^2}dW(t)$, and the drift-diffusion structure ensures $X(t) \in [-1,1]$ and also robotic navigation models of the form $dX(t) = -\lambda\cos^{-1}(X(t))dt + \sigma X(t)dW(t)$ leverage, ensuring operational safety and these frameworks bridge theoretical guarantees with practical constraints in the most applications , distinguishes this approach is its ability to combine mathematical stochastic constraints with realistic physical and biological models, allowing it to be used in a variety of fields (such as models of cell growth or in neural propagation), finance (in characterizing market boundaries or portfolio behavior within limited risk frameworks) and engineering (particularly in mechatronic systems and precise robotic control) and thus, the stochastic models are not confined to theoretical mathematical formulations, but rather become effective tools for ensuring safety under real and changing operational conditions.

Future Directions:

For the Fractional SDE: Replace the standard Brownian motion $W(t)$ by fractional Brownian motion $W^H(t)$ (Hurst index $H \ne 1/2$) in systems like:

$$dX(t) = -\lambda\arctan(X(t))dt + \sigma X(t)dW^H(t).$$

The long-range dependence of the fractional $W^H(t)$ alters moment dynamics, necessitating tools from rough path theory or Malliavin calculus.

Multi-Dimensional Systems: For coupled SDEs like:

$$\begin{aligned} dX(t) &= -\kappa\sin^{-1}(Y(t))dt + \sigma_1 dW_1(t), \\ dY(t) &= -\kappa\cos^{-1}(X(t))dt + \sigma_2 dW_2(t), \end{aligned}$$

cross-moments $\mathbb{E}[X^k Y^m]$ require tensorized Lyapunov functions $V(X,Y) = \left(\sin^{-1}(Y)\right)^2 + \left(\cos^{-1}(X)\right)^2$ and advanced closure schemes.

Machine Learning for Moment Closure: Train neural networks to approximate unresolved moments $\mathbb{E}[X^p]$ from low-order data $\{\mathbb{E}[X], \mathbb{E}[X^2]\}$, for example a feedforward network $\mathcal{N}$ could learn, where θ are weights optimized via stochastic gradient descent on simulated SDE trajectories [14], Physics-informed architectures could further embed Itô's lemma into the loss function:

$$\mathbb{E}[X^3] \approx \mathcal{N}(\mathbb{E}[X], \mathbb{E}[X^2]; \theta),$$

$$\mathcal{L}(\theta) = \| \frac{d}{dt}\mathbb{E}[X^2] - (2\mathbb{E}[XA(X)] + \mathbb{E}[B^2(X)]) \|^2.$$

These advancements will expand the scope of tractable nonlinear SDEs, enabling predictive modeling in systems where classical assumptions fail to capture bounded yet noise-driven dynamics.

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Conflict of interest

None.

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