Al-Rafidain Journal of Computer Sciences and Mathematics (RJCSM)

www.csmj.uomosul.edu.iq

The Structure of Tripotent Graph of Certain Commutative Rings

Parween O. Ali[1] and Shaymaa S. Essa[2]

[1,2] College of Education, University of Duhok, Duhok, Kurdistan Region-Iraq
Email: parweenomar97@gmail.com[1] and shaymaa.essa@uod.ac[2]

Article information

Article history:
Received 3 March, 2026
Revised 3 May,2026
Accepted 15 May, 2026
Published 25 June, 2026

Keywords:
tri-potent element,
tri-potent graph,
diameter,
independent domination number,
metric dimension.

Correspondence:
Shaymaa S. Essa
Email: shaymaa.essa@uod.ac

Abstract

In this paper, we characterize the texture of tripotent graph of the ring, which is associated by tripotent element of a ring such that, for every two distinct vertices; s and t then, their adjacent if and only if $(s+t)^{3}=s+t$. We establish several structural graphical properties of tripotent graph of local ring when it is a path graph and it is spanning subgraph with unit graph and clean graph. Also, we obtain that, the tripotent graph is Hamiltonian graph but doesn't Eulerian. Moreover, we investigate the concept of tripotent graph of a commutative ring R where $R \cong Z_{2r}$ and $R \cong Z_{3r}$ for r is a prime integer as $diam(TP(R))$, $m\,(TP(R))$ and $\omega(TP(R))$ which equals to $\chi(TP(R))$. Furthermore, we prove that the tripotent graph of commutative ring is 4-partite graph and $\gamma_{ind}TP(R)=r$, for $R \cong Z_{2r}$. Finally, we are concerned about the metric dimension of tripotent graph of some cases of commutative rings, especially; where tripotent graphs are planar and nonplanar.

DOI: 10.33899/rjcsm.v20i1.60672, ©Authors, 2026, College of Computer Science and Mathematics, University of Mosul, Iraq.

1. Introduction

Graph theory and ring theory are fundamental branches of mathematics, and the interaction between them has led to many useful approaches for studying algebraic textures through graphs. In recent years, several types of graphs associated with rings have been introduced in the literature, where the vertices correspond to elements of a ring and adjacency is determined by certain algebraic conditions. Among these constructions, the unit graph represents each element of a ring as a vertex, and two distinct vertices s and t are adjacent whenever their sum $s+t$ is a unit element of the ring [3]. Another related construction is the idempotent graph, in which two vertices are connected whenever their sum is an idempotent element [13]. In a similar direction, the clean graph has also been studied, where adjacency occurs whenever the sum of two elements is either a unit or an idempotent element [12]. More recently, the concept of the tripotent graph of a commutative ring has been introduced by Essa and Ali [5], as the vertex set consists of all elements of the ring R, and two distinct vertices s and t are adjacent whenever the element $s+t$ is tripotent; that is, it satisfies the condition $(s+t)^{3}=s+t$, and its studies such graphs provides insight into how algebraic properties of rings influence the texture of the corresponding graphs [5].

Many works in the literature focus on analyzing structural properties of graphs associated with rings, such as connectivity, diameter, clique number, chromatic number, and planarity. These investigations contribute to a deeper understanding of the relationship between algebraic properties of rings and graph-theoretic parameters.

Throughout this paper, all rings are assumed to be commutative with identity 1, and we study the texture of the tripotent graph of certain commutative rings. In particular, we demonstrate the size of the tripotent graph and show that $\omega\big(TP(R)\big)=\chi\big(TP(R)\big)=4$ or 6, depending on the texture of the sitting ring. Moreover, it concernes the tripotent graph of a ring which forms a 4-partite graph. Moreover, we investigate several graph invariants such as the diameter, clique number, chromatic number, independent domination number, and metric dimension for specific classes of rings.

2. Preliminaries

This section presents several definitions and basic concepts from graph theory and ring theory that will be used throughout the paper. For undefined terminology, the reader may refer to standard references in these areas. Let a graph $G = (V, E)$ be a simple graph where $V(G)$ denotes the set of vertices and $E(G)$ denotes the set of edges . The order (size) of G is represents to the number of vertices(edges), denoted by $n; m$, respectively. Two distinct vertices u and v are said to be adjacent if there exists an edge joining them . A path in a graph G is a sequence of vertices where each consecutive pair is connected by an edge, and neither vertices nor edges are repeated. A graph G is connected if there is a path between every pair of vertices, otherwise, it is disconnected. The distance between two vertices is defined as the length of the shortest path connecting them, and the diameter of a connected graph is the maximum distance between any pair of vertices and denotes by $diam(G)$ [7]. The degree of a vertex x, represents as $deg(x)$; is the number of edges incident to x , we have a cycle is a graph in which each vertex has degree two. The girth of a graph G, denoted by $gir(G)$, is the length of its shortest cycle, and if G has no cycle, the girth is considered infinite. The open neighborhood of a vertex x denoted by $N(x)$ which is the set of all vertices in G that are adjacent to x, otherwise; the closed neighborhood of x denoted by $N[x]$, is defined as the union of the open neighborhood of x and the vertex x itself . The clique number of a graph G, is the maximum number of vertices in any clique of G, and denotes by $\omega(G)$. A graph G is said to be Eulerian if it contains a closed trail, that traverses every edge of the graph exactly once. A graph G is said to be Hamiltonian if it contains a cycle, that passes each vertex of G exactly once. A graph in which all vertices are pairwise adjacent is called a complete graph and denotes by K_n, where n is the number of vertices. A graph G is called a k-partite graph if its vertex set $V(G)$ can be partitioned into k mutually disjoint non-empty subsets $V_1, V_2, \dots, V_k$, such that no edge connects two vertices within the same subset . A spanning subgraph of G is a subgraph containing all the vertices of G. A planar graph is a graph that can be embedded in the plane without edge crossings. The chromatic number of a graph G, denoted by $\chi(G)$, is the smallest number of distinct colors required to color the vertices of the graph. An independent dominating set in a graph, denoted as $\gamma(G)$ is a set of vertices that is both independent and dominating, that is; no two vertices in the set are adjacent, and every other vertex in the graph is either in the set or is adjacent to at least one vertex in the set [9]. For a simple connected graph G with vertex set V(G), let three vertices $r, s, t \in V(G)$, if the shortest path distance $d(r, t) \neq d(s, t)$, then the vertex t is said to resolve the vertices r and s and a subset $S \subseteq V(G)$ is said to be a resolving set of G if every pair of distinct vertices in G can be distinguished by at least one vertex from S [6]. The metric dimension of G, denoted by $M_{dim}(G)$, considered as the minimum cardinality of such a resolving set [6]. A ring R is said to be local ring if it has a unique maximal ideal of the ring. A ring R is called a reduced ring if it has no nonzero nilpotent elements. An element s in a ring R with identity is called a unit if there exists an element $u \in R$, such that $us = su = 1$, the group of units of R, is denoted by $U(R)$, see [4]. An element s of a ring R is said to be idempotent if satisfies the condition $s^2 = s$, and the set of all idempotent elements denoted by $Id(R)$. An element t in R is called tripotent if $t^3 = t$, and the set of all tripotent elements is denoted by Tri(R). Consider the tripotent elements are precisely $\{0,1,-1\}$ in $\mathbb{Z}$, and every idempotent is necessarily tripotent [10].

3. Algebraic Structure of Tripotent Element

This section is inspected the tripotent element of commutative ring in several finite settings, with particular emphasis on the case where the ring is local and reduced.

Lemma 3.1. For local ring R, then every nonzero tripotent element in R is a unit.

Proof. Since R is a local ring, then it has a unique maximal ideal such (p) . Now let $t \in Tri(R)$, then $t^3 = t$,implies that $t(t^2 - 1) = 0$, implies that either t or $1 - t$ has a unit, this lead $t = 0$ which contracts.

Lemma 3.2. Let $R \cong Z_{pr}$, where p and r are two distinct primes. Then:

i. If $p = 2$, then $Tri(R) = \{0,1,r,r+1,r-1,2r-1\}$.
ii. If $p = 3$, then $Tri(R) = \{0,1,r,2r,r+1,r-1,2r-1,2r+1,3r-1\}$.

Proof. i) Since $R \cong Z_{2r}$, where r is a prime integer. It possess $Z_{2r} \cong Z_2 \times Z_r$, and since $gcd(2, r) = 1$, then for every $s \in Tri(R)$ satisfies $s^3 \equiv s(mod\ 2r)$. Thus, by CRT, we can solve the following equations:

$s^3 \equiv s(mod\ 2)$…..(1)
$s^3 \equiv s(mod\ r)$…..(2)

Since, by definition of tripotent element, it possesses $s \equiv 0(mod\ 2)$ and $s \equiv 1(mod\ 2)$. For (1).

Also , the tri-potent element of (2) is described by $s \equiv 0(mod\ r)$, $s \equiv 1(mod\ r)$, and $s \equiv r - 1(mod\ r)$.

Now, we concern the following cases:

Case1. If $s \equiv 0(mod\ r)$, then $s = rk$, for $k \in \mathbb{Z}^+$. Since by (1), we need to change $k \in Tri(\mathbb{Z}_2)$. So we obtain $s \equiv 0(mod\ 2r)$ and $s \equiv r(mod\ 2r)$.

Case2. If $s \equiv 1(mod\ r)$, then $s = 1 + rk$, for $k \in \mathbb{Z}^+$, since by (1) we take $k \in Tri(\mathbb{Z}_2)$. That is ; $s \equiv 1(mod\ 2r)$ and $s \equiv r + 1(mod\ 2r)$.

Case3. If $s \equiv r - 1(mod\ r)$, then $s = (r - 1) + rk$, for some $k \in \mathbb{Z}^+$. By changing $k \in \mathbb{Z}^+$ to $k \in Tri(\mathbb{Z}_2)$. We get $s \equiv r - 1(mod\ 2r)$ and ; $s \equiv 2r - 1(mod\ 2r)$.

Hence $Tri(R) = \{0,1, r, r + 1, r - 1,2r - 1\}$.

ii). Now let $R \cong Z_{3r}$, where r is a prime integer, then $Z_{3r} \cong Z_3 \times Z_r$, and $gcd(3, r) = 1$, then for every $s \in Tri(R)$, and it possesses $s^3 \equiv s(mod\ 3r)$. Thus, by Chinese Remainder Theorem, we can solve the following equations:

$s^3 \equiv s(mod\ 3)$…..(1)

$s^3 \equiv s(mod\ r)$…..(2)

Since, by definition of tripotent element, it possesses $s \equiv 0(mod\ 3)$ and $s \equiv 1(mod\ 3)$. $s \equiv 2(mod\ 3)$, For (1).

Also , the tri-potent element of (2) is described by $s \equiv 0(mod\ r)$, $s \equiv 1(mod\ r)$, and $s \equiv r - 1(mod\ r)$. Now, we concern the following cases:

Case1. If $s \equiv 0(mod\ r)$, then $s = rk$, for $k \in \mathbb{Z}^+$. Since by (1), we need to change $k \in Tri(\mathbb{Z}_3)$. So we obtain $s \equiv 0(mod\ 3r)$, $s \equiv r(mod\ 3r)$ and $s \equiv 2r(mod\ 3r)$.

Case2. If $s \equiv 1(mod\ r)$, then $s = 1 + rk$, for $k \in \mathbb{Z}^+$, since by (1) we take $k \in Tri(\mathbb{Z}_3)$. That is ; $s \equiv 1(mod\ 3r)$, $s \equiv r + 1(mod\ 3r)$,and $s \equiv 2r + 1(mod\ 3r)$.

Case3. If $s \equiv r - 1(mod\ r)$, then $s = (r - 1) + rk$, for some $k \in \mathbb{Z}^+$. By changing $k \in \mathbb{Z}^+$ to $k \in Tri(\mathbb{Z}_3)$. We get $s \equiv r - 1(mod\ 3r), s \equiv 2r - 1(mod\ 3r)$ and $s \equiv 3r - 1(mod\ 3r)$. Hence

$Tri(R) = \{0,1, r, 2r, r + 1, r - 1,2r - 1,2r + 1, 3r - 1\}$.

Lemma 3.3. Let $R \cong Z_{pr}$, where p and r are two distinct primes. Then $\mathcal{M}(R) = \{t: 2t \in Tri(R)\}$, such that:

i.If $p = 2$, then $|\mathcal{M}(R)| = 6$.

ii.If $p = 3$, then $|\mathcal{M}(R)| = 9$.

Proof. Let $R \cong Z_{2r}$, where r is a prime integer. For every $s \in Tri(R)$, then $s^3 \equiv s(mod\ 2r)$, we seek to find those elements such t where $2t \in Tri(R)$, since $R \cong Z_2 \times Z_r$, and we use Chinese Remainder Theorem to find the value of t, that is ;

$$t \equiv a(mod\ 2)$$
$$t \equiv b(mod\ r)$$

Now, $t \equiv b(mod\ r)$, then $t = b + rk$ where $k \in \mathbb{Z}^+$, since by [5, Lemma 4.1(i)], we take $k \in M(\mathbb{Z}_2)$. Thus,

If $k = 0$, it possesses $t \equiv b(mod\ 2r)$.

If $k = 1$, it possesses $t \equiv b + r(mod\ 2r)$.

It possesses the following cases:

Case 1. If $t \equiv 0(mod\ r)$, then $t \equiv 0(mod\ 2r)$, and $t \equiv r(mod\ 2r)$.

Case 2. If $t \equiv \frac{r+1}{2}(mod\ r)$, then If $t \equiv \frac{r+1}{2}(mod\ 2r)$, and $t \equiv \frac{3r+1}{2}(mod\ 2r)$.

Case 3. If $t \equiv \frac{r-1}{2}(mod\ r)$, then If $t \equiv \frac{r-1}{2}(mod\ 2r)$, and $t \equiv \frac{3r-1}{2}(mod\ 2r)$.

Thus, $M(R) = \left\{0, r, \frac{r+1}{2}, \frac{r-1}{2} \frac{3r+1}{2}, \frac{3r-1}{2}\right\}, and\ |M(R)| = 6$.

ii) Let $R \cong Z_{3r}$, where r is a prime integer. For every $s \in Tri(R)$, then $s^3 \equiv s(mod\ 3r)$, we seek to find those elements such t where $2t \in Tri(R)$, since $R \cong Z_3 \times Z_r$, and we use Chinese Remainder Theorem to find the value of t, that is ; $t \equiv a(mod\ 3)$

$t \equiv b(mod\ r)$

Now, $t \equiv b(mod\ r)$, then $t = b + rk$ where $k \in \mathbb{Z}^+$, since by [5, Lemma 4.1(ii)], we take $k \in M(\mathbb{Z}_3)$. Thus,

If $k = 0$, it possesses $t \equiv b(mod\ 3r)$.

If $k = 1$, it possesses $t \equiv b + r(mod\ 3r)$.

If $k = 2$, it possesses $t \equiv b + 2r(mod\ 3r)$.

It possesses the following cases:

Case 1. If $t \equiv 0(mod\ r)$, then $t \equiv 0(mod\ 3r)$, $t \equiv r(mod\ 3r)$, and $t \equiv 2r(mod\ 3r)$.

Case 2. If $t \equiv \frac{r+1}{2}(mod\ r)$, then $t \equiv \frac{r+1}{2}(mod\ 3r)$, $t \equiv \frac{3r+1}{2}(mod\ 3r)$, and $t \equiv \frac{5r+1}{2}(mod\ 3r)$.

Case 3 If $t \equiv \frac{r-1}{2}(mod\ r)$, then $t \equiv \frac{r-1}{2}(mod\ 3r)$, $t \equiv \frac{3r-1}{2}(mod\ 3r)$, and $t \equiv \frac{5r-1}{2}(mod\ 3r)$.

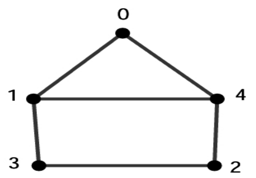
Thus, $M(R) = \left\{0, r, 2r, \frac{r+1}{2}, \frac{r-1}{2}, \frac{3r+1}{2}, \frac{3r-1}{2}, \frac{5r+1}{2}, \frac{5r-1}{2}\right\}$, and $|M(R)| = 9$.

4. Structure of Tripotent graph of Commutative Rings

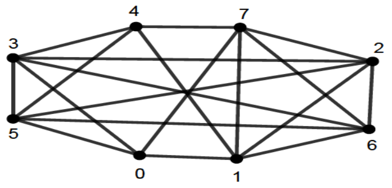
In this section, we explore the interplay between algebra and graph theory, which has become an important theme in modern mathematics. Graphs offer an effective framework for representing and analyzing the elements of rings. We begin by viewing the definition of the tripotent graph of a commutative ring, followed by several illustrative examples. These examples highlight the distinctive structural features and properties that arise in certain classes of local and reduced rings.

Definition 4.1. [5] For a ring R, define the tripotent graph as a graph with the set of vertices as the element of R, and for every two elements s and t in R they are adjacent if $s + t$ is a tripotent element and this graph denoted by $TP(R)$.

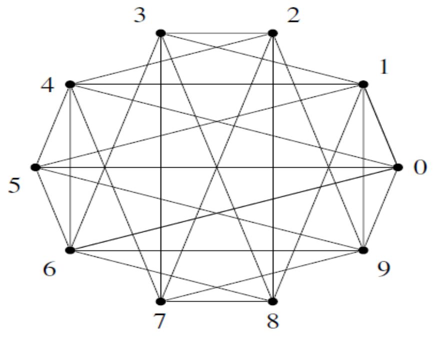
Example 4.2. In **Figure1**, we provide some examples of tripotent graph of commutative rings.



1.a $TP(Z_5)$



1.b $TP(Z_8)$



1.c $TP(Z_{10})$

Figure 1. Tripotent Graph of Commutative Rings.

Remark 4.3. Consider the **Table 1**, which presents the properties of $TP(R)$.

Table 1. Properties of $TP(R)$.

V($TP(R)$)	Type of ring R	$TP(R)$
2	Z_2	K_2
3	Z_3	C_3
4	Z_4 or $Z_2[x]/(x^2)$	$K_1 + P_3$
5	Z_5	Figure 1.a
6	Z_6	K_6
8	Z_8	Figure 1.b
10	Z_{10}	Figure 1.c

Proposition 4.4. Let $R \cong Z_w$, where $w = p^\alpha$, $\alpha \in \mathbb{Z}^+$ with $p \geq 3$, and R has only nonzero tripotent elements, then $TP(R)$ is a path graph.

Proof. Let $R \cong Z_w$, where $w = p^\alpha$, $\alpha \in \mathbb{Z}^+$ with $p \geq 3$, then $Tri(R) = \{1, -1\}$. Now for two distinct vertices $t_1, t_2 \in TP(R)$, then $t_1 + t_2 \in Tri(R)$, either $t_1 + t_2 = 1$ or $t_1 + t_2 = -1$, therefore we obtain the path $(w+1)/2 - \cdots -(w-2) - 1 - 0 - (w-1) - 2 - \cdots - (w-1)/2$, such that $(w+1)/2 + (w-1)/2 = 0$, and since $0 \notin Tri(R)$, it follows that $(w+1)/2 + (w-1)/2 \notin TP(R)$. Thus $TP(R)$ is a path graph.

Proposition 4.5. For local ring R with $(p)^i = (0)$, where p is a prime integer. If R has nonzero tripotent elements, then $TP(R)$ is a spanning subgraph of $G_u(R)$.

Proof. For two distinct vertices $t_1, t_2 \in TP(R)$, it possesses $t_1 + t_2 = t \in Tri(R)$, by Lemma 3.1, yields $t^3 = t \in U(R)$. Thus $t_1 + t_2 \in U(R)$. Hence $TP(R)$ is a spanning subgraph of $U_G(R)$.

Proposition 4.6. For local ring R with $(p)^i = (0)$, where p is a prime integer. If R has nonzero tripotent elements, then $TP(R)$ is a spanning subgraph of $Cl_G(R)$.

Proof. For two distinct vertices $t_1, t_2 \in TP(R)$, it possesses $t_1 + t_2 \in Tri(R)$, since every idempotent element is also a tripotent element, therefore $t_1 + t_2 \in Id(R)$. Also, by Lemma 3.1, we obtain $t_1 + t_2 \in U(R)$. Hence, $t_1 + t_2 \in Id(R) \cup U(R)$. Thus, $t_1 + t_2 \in Cl_G(R)$. Therefore, $TP(R)$ is a spanning subgraph of $Cl_G(R)$.

Proposition 4.7. For local ring R, then $TP(R)$ is a Hamiltonian graph.

Proof. Suppose that $TP(R)$ is a connected graph, for any two distinct vertices t_1 and t_2, such that $t_1 + t_2 \in Tri(R)$. Now we need to find a cycle which contains every vertices of $TP(R)$. Consider the cycle such $0 - (r-1) - 2 - (r-3) - 4 - (r-5) - 6 - \cdots - (1) - 0$, which represents to C_r where $Char(R) = r$. Thus is Hamiltonian graph.

Proposition 4.8. For a ring R, then $TP(R)$ is not Eulerian graph.

Proof: Let $TP(R)$ be a connected graph and let t_i are vertices in $TP(R)$. If $2t_i \in Tri(R)$, then $\deg(t_i) = |Tri(R)| - 1$, by [5, Lemma 3.5], that is $\deg(t_i)$ is an even. However, if $2t_i \notin Tri(R)$, then $\deg(t_i) = |Tri(R)|$, and its odd. Hence $TP(R)$ is not Eulerian graph.

Theorem 4.9. For local ring R, then:

i. If $TP(R)$ is planar, then $M_{dim}\,(TP(R)) = 2$.

ii. If $R \cong Z_r$, where $r = 2^{\alpha}, \alpha \geq 4$. Then $M_{dim}\ (TP(R)) = diam(TP(R)) + 1$.

Proof. i. Let $TP(R)$ be a tripotent graph of a ring R , where $R \cong Z_{p^{\alpha}}$, $with\, p \geq 3\ and\ \alpha \in \mathbb{Z}^{+}$. Suppose that $S = \{s_1, s_2\}$ be a resolving set of $V\big(TP(R)\big)$ such that $s_1 \in V_2$ and $s_2 \in V_3$, where $V_2 = \left\{1,2,..,\frac{r-1}{2}\right\}$ and $V_3 = \left\{\frac{r+1}{2},\dots,r-1\right\}$, by according to $[5, Theorem\ 4.3\ ]$. Now, for every two distinct vertices such t_i and t_j in $V\big(TP(R)\big)$, then $d(t_i, s_1) \neq d\big(t_j, s_1\big)$ and $d(t_i, s_2) \neq d\big(t_j, s_2\big)$. Thus $\psi(t_i|S) \neq \psi\big(t_j|S\big)$. Next, we demonstrate to show that S is a minimum resolving set. Assume that $S = \{s\}$, then there exist two distinct vertices as t_i and t_j in $V\big(TP(R)\big)$, such that $d(t_i, s) = d\big(t_j, s\big)$, implies that $\psi(t_i|S) = \psi\big(t_j|S\big)$. Hence $S = \{s_1, s_2\}$ is a minimum resolving set of $TP(R)$. Thus $|S| = 2 = M_{dim}\ (TP(R))$.

ii. Let $TP(R)$ be a tripotent graph of a ring R , where $R \cong Z_{2^{\alpha}}$ and $\alpha \geq 4$. Suppose the resolving set $S \subseteq V\big(TP(R)\big)$ and $S = \{s_0, s_1, \dots, s_k\}$, where $0 \leq k \leq diam(TP(R))$. Now, for every two distinct vertices t_i and t_j in $V\big(TP(R)\big)$, then $\psi(t_i|S) \neq \psi\big(t_j|S\big)$, therefore S is a resolving set of $TP(R)$. Next ,it possesses to show that S is a minimum resolving set of $TP(R)$, assume that $1 \leq k \leq diam(TP(R))$, see [5, Theorem 3.7]., then there exists $i \neq j$ which is $t_i = t_j$ such that $i = 2^{\alpha-2}$ and $j = 3 \cdot 2^{\alpha-2}$. Thus $\psi(t_i|S) = \psi\big(t_j|S\big)$. Hence $S = \{0,1,2,\dots, diam(TP(R))\}$ is a minimum resolving set of $TP(R)$ with $|S| = diam(TP(R)) + 1$ and $M_{\dim(TP(R))} = diam(TP(R)) + 1$.

Remark 4.10. If $R \cong Z_r$, where $r = 2^{\alpha}$, $\alpha = 3$. Then $\dim(TP(R)) = 2^{\alpha-1} = 4$, as the Example.

Example 4.11. Suppose that $S = \{0,1,2,3\}$ is a resolving set and the graph $TP(R)$ shown in **Figure 1.b** is given. We compute the metric representation of all vertices with respect to S:

$\psi(t_i \backslash S) = \big(d(t_i, 0), d(t_i, 1), (t_i, 2), d(t_i, 3)\big)$

$\psi(0 \backslash s_i) = (0,1,2,1),$

$\psi(1 \backslash s_i) = (1,0,1,2),$

$\psi(2 \backslash s_i) = (2,1,0,1),$

$\psi(3 \backslash s_i) = (1,2,1,0),$

$\psi(4 \backslash s_i) = (2,1,2,1),$

$\psi(5 \backslash s_i) = (1,2,1,1),$

$\psi(6 \backslash s_i) = (2,1,1,1),$

$\psi(7 \backslash s_i) = (1,1,1,2).$

We see that $\psi(t_i \backslash S) \neq \psi\big(t_j \backslash S\big)$ for all $i \neq j$, and hence S is a minimum resolving set. Therefore, $M_{dim}\ (TP(R)) = 2^{\alpha-1}$.

5. Structure of Tripotent graph of Reduced Rings

In this section, we study the tripotent graph associated with a finite reduced ring, especially when R isomorphic direct product two fields. We first determine the order of the tripotent graph and show that both its clique number and chromatic number, which equals 4 or 6, are governed by the underlying texture of the ring. We then demonstrate that the tri-potent graph is a 4-partite graph and computes its independent domination number, which is equal to $r - 2$, where r is a prime number greater than 3. Finally, we analyze the metric dimension of the tripotent graph and prove that it is equal to r for any prime $r > 3$.

Proposition 5.1. Let R be a ring, and let $R \cong Z_{pr}$, where p and r are two distinct primes with $p < r$. Then

$$m\ (TP(R)) = \begin{cases} 3(r-1), & if\ R \cong Z_{2r}, where\ r > 3\ is\ a\ prime\ integer. \\ \dfrac{9(r-1)}{2} & if\ R \cong Z_{3r}, for\ r\ is\ a\ prime\ integer. \end{cases}$$

Proof. Since every edge in a graph is incident to exactly two vertices, the total number of edges in $TP(R)$ can be expressed as $m\big(TP(R)\big) = \sum_{i=1}^{r} deg(v_i)/2$, then by [5, Lemma 3.5], it possesses, $deg(v_i) = |Tri(R)| - 1$ or $deg(v_i) = |Tri(R)|, that\ is$:

$$m\big(TP(R)\big) = \left(\sum_{i=1}^{k} (|Tri(R)| - 1) + \sum_{i=k+1}^{|R|} (|Tri(R)|)\right)\Big/2$$

Now, put $k = |\mathcal{M}(R)|$, which defines in Lemma 3.3, it possesses two following cases:

i.For $|Tri(R)| = 6$, and $|R| = r$, so it possesses

$$m(TP(R)) = \left(\sum_{i=1}^{6} (6-1) + \sum_{i=7}^{r} 6\right)\Big/2$$

$$= \left(\sum_{i=1}^{6} 6 - \sum_{i=1}^{6} 1 + \sum_{i=7}^{r} 6\right)\Big/2$$

$$= \left(\sum_{i=1}^{r} 6 - \sum_{i=1}^{6} 1\right)\Big/2$$

Thus, $m(TP(R)) = \frac{6r-6}{2} = 3(r-1)$.

ii.If $k = 9$, then $|Tri(R)| = 9, so$

$$m(TP(R)) = \left(\sum_{i=1}^{9} (9-1) + \sum_{i=10}^{r} 9\right)\Big/2$$

$$= \left(\sum_{i=1}^{9} 9 - \sum_{i=1}^{9} 1 + \sum_{i=10}^{r} 9\right)\Big/2$$

$= \left(\sum_{i=1}^{r} 9 - \sum_{i=1}^{9} 1\right)/2$

Thus, $m(TP(R)) = \frac{9(r-1)}{2}$.

Theorem 5.2 For a ring R, then $TP(R)$ is connected and $diam\big(TP(R)\big)$

$$= \begin{cases} \left\lceil \dfrac{2r}{\lceil \mathcal{M}(R) \rceil - 1} \right\rceil & if\ R \cong Z_{2r}, where\ r > 2\ is\ a\ prime\ integer. \\ \left\lceil \dfrac{3r}{\lceil \mathcal{M}(R) \rceil - 1} \right\rceil & if\ R \cong Z_{3r}, where\ r > 3\ is\ a\ prime\ integer. \end{cases}$$

Proof. Let $t_1$and t_2 be two distinct vertices in $TP(R)$. If $t_1 + t_2$ equals only the trivial tripotent element of R, then $TP(R)$ is disconnected, since at least one component—such as the subgraph where 0 is adjacent to 1. Conversely, if $t_1 + t_2 \in Tri(R)$, then there exists a path connecting 0 and $t_1$of the form $0 - y_1 - y_2 - \cdots - y_n - t_1$, for some vertices y_i, where the path length is even and satisfies $0 + t_1 \in Tri(R)$. Hence, $TP(R)$ is connected.

Now, consider the previous cases to determine $diam\big(TP(R)\big)$ as

Case 1. Let $t_1, t_2 \in R$.. If $t_1 + t_2 \in Tri(R)$, then $t_1$and $t_2$are adjacent, which gives a direct path $t_1 - t_2$. Hence,

$$d(t_1, t_2) = 1,$$

and therefore, $diam\big(TP(R)\big) = 1$.

Case 2. Let $t_1, t_2 \in R$. If $t_1 + t_2 \notin Tri(R)$, then there exist elements $a_1, a_2, \dots, a_n \in R$ such that a path of the form $t_1 - a_1 - t_2$ exists, implying that $diam\big(TP(R)\big) = 2$. However, if $t_1 + a_1 \notin Tri(R)$, then an additional vertex $a_i \in R$ can be found to extend the path, giving $a_1 - a_2, - \cdots - a_n - t_2$. Thus, to determine the diameter precisely, we proceed to examine the following specific subcases.

i. If $R \cong Z_{2r}$, then $d(0, \frac{r-1}{2}) = d(0, \frac{r+1}{2}) = d(0, r - \frac{r-1}{2}) = d(0, r - \frac{r+1}{2})$, that is $diam(TP(R)) = \left\lceil \frac{2r}{\lceil \mathcal{M}(R) \rceil - 1} \right\rceil$.

ii. If $R \cong Z_{3r}$, then $d(0, \frac{r-1}{2}) = d(0, \frac{r+1}{2}) = d(0, \frac{3r-1}{2}) = d(0, \frac{3r+1}{2}) = d(0, r - \frac{r-1}{2}) = d(0, r - \frac{r+1}{2})$, that is $diam(TP(R)) = \left\lceil \frac{3r}{\lceil \mathcal{M}(R) \rceil - 1} \right\rceil$.

Theorem 5.3. If $R \cong Z_{pr}$, where p and r are two distinct primes, and $p < r$. Then:

i. If $p = 2$, then $\omega(TP(R)) = \chi(TP(R)) = 4$.

ii. If $p = 3$, then $\omega(TP(R)) = \chi(TP(R)) = 6$.

Proof: i) By Lemma 3.2(i), suppose that $t \in \mathcal{M}\big(TP(R)\big)$, where $R \cong Z_{2r}$ and r is a prime integer. Take $N[t]$ as an induced subgraph of $TP(R)$ and put $t = 0$, so we get

$$N[0] = \{0, 1, r-1, r, r+1, 2r-1\}$$

Assume that $TP_{N[0]}(R) = \{0, 1, r, r+1, 2r-1\}$ represent the vertices of an induced subgraph of $TP(R)$ which induces from $N[0]$. By [5, Lemma 3.5] ,it possesses $\deg(a) = |Tri(R)|\ or\ |Tri(R)| - 1$, for every $a \in V\big(TP(R)\big)$, that is; $\deg(a) = 6\ or\ 5$. Particularly, $\deg(a) = 5\ or\ 4$, for every $a \in V\left(TP_{N[0]}(R)\right)$. Then at least for 1 and $r+1$ in $TP_{N[0]}(R)$, we obtain $d(1, r+1) = 2$, that is;

$1 + (r+1) = r + 2 \notin Tri(R)$, so $deg(1) = deg(r+1) = 3$. Thus $TP(R)$ does not contain K_5 as a subgraph of $TP(R)$. Also, assume $TP_{N[0]}(R) = \{0, 1, r-1, r, 2r-1\}$ as a vertex of an induced subgraph of $TP(R)$ which induced from $N[0]$. By similar way of above, it possesses $d(r-1, 2r-1) = 2$, such that $(r-1) + (2r-1) = r - 2 \notin Tri(R)$.

Hence no K_5 as a subgraph contains $TP(R)$, furthermore, $\omega(TP(R)) = \chi(TP(R)) = 4$.

ii) Let $t \notin Tri(R)\ \ such\ that\ t \in \mathcal{M}\big(TP(R)\big)$, by Lemma 3.2(ii), it possesses $\deg(t) = 8$. We take $N[t]$ as an induced subgraph of $TP(R)$. Now we show that K_7 does not contain in $TP(R)$. Suppose the subgraph $TP_{N[0]}(R)$ contain only 7 vertices, then $V\big(TP_{N[t]}(R)\big) = \{t, y_i; 1 \le i \le 6\}$, so we concern the following cases;

Case1. If for all y_i and y_j in $V\left(TP_{N[t]}(R)\right)$ and $y_i + y_j \in Tri(R)$, then $d\big(y_i, y_j\big) = 1$. That is ; $TP(R)$ contains K_7.

Case 2. If for all y_i and y_j in $V\left(TP_{N[t]}(R)\right)$ and $y_i + y_j \notin Tri(R)$, then $d\big(y_i, y_j\big) = 2$. That is ; $TP(R)$ does not contain K_7. As a result , at least there is case 2, shows that ; $TP(R)$ does not contain K_7. Thus $TP(R)$ contains K_6. Furthermore, $\omega(TP(R)) = \chi(TP(R)) = 6$.

Remark 5.4. Consider that if $t \in Tri(R)\ \ and\ t \in \mathcal{M}\big(TP(R)\big)$, then the result of Theorem 5.3, is not verified, because $V\left(TP_{N[0]}(R)\right)$ doesn't induce a complete subgraph K_6 see Example 5.5.

Example 5.5. $R \cong Z_{3r}$ where $r = 5$, suppose that $t = 0$ and $2(0) \in \mathcal{M}\big(TP(R)\big)$, and $0 \in Tri(R)$, so $N[0]$={0,1,4,5,6,9,10,11,14}, that is ; by [5, Lemma 3.5], $\deg(t) = 8$.

Now, take $V\left(TP_{N[0]}(R)\right)$ as a vertex of an induced subgraph of $TP(R)$, then it possesses $\binom{8}{5}$ cases to choose the set of vertices $V\left(TP_{N[0]}(R)\right)$. Now take:

$V\left(TP_{N[0]}(R)\right) = \{0,1,4,5,6,9\}$

That $d(4,9) = 2$ such that $4 + 9 = 13 \notin Tri(R)$, so $TP_{N[0]}(R)$ does not contain K_6.

Theorem 5.6. Let $R \cong Z_{2r}$ where $r > 3$ is a prime integer. Then $TP(R)$ is a 4-partite graph.

Proof. Let $R \cong Z_{2r}$ where $r > 3$ is a prime integer, and suppose the partitions as follow:

$P_1 = \{v_i : 1 \le i \le \alpha\},\, where\ \alpha = \frac{r-1}{2}$

$P_2 = \{v_j : \alpha + 1 \le j \le 2\alpha\}$,

$P_3 = \{v_k : 2\alpha + 2 \le k \le 3\alpha\}$,

$P_4 = \{v_l : 3\alpha + 3 \le l \le 4\alpha + 1\}$,

$P_5 = \{0, 3\alpha + 2\}$,

$P_6 = \{2\alpha + 1, 3\alpha + 1\}$.

Since every P_i, for all i is an independent set, because every two distinct vertices t and y in P_i, then $t + y \notin Tri(R)$. Therefore $\cup P_i$ where $1 \le i \le 6$ is equal to $V\left(TP(R)\right)$ and P_i, $1 \le i \le 6$ are disjoint.

Now, by Theorem 5.3, we get $\omega(TP(R)) = \chi(TP(R)) = 4$, so the minimum partite of $TP(R)$ is equal tom 4 as the following:

Let $V_1 = (P_1 \cup P_3)$, since every two distinct vertices of V_1 is adjacent to P_i for $i = 2,4,5,6$ and let $V_2 = (P_2 \cup P_4)$, because every two distinct vertices of V_2 is adjacent to P_i for $i = 1,3,5,6$, also let $V_3 = P_5$ and $V_4 = P_6$, for the same reason. That is; $TP(R)$ is a 4-partite graph.

Theorem 5.7. Let $R \cong Z_{2r}$ where $r > 3$ is a prime integer. Then $\gamma_{ind}\left(TP(R)\right) = r - 2$.

Proof. Let $R \cong Z_{2r}$ where $r > 3$ is a prime integer. By Theorem 5.3, consider that

$$D = (P_1 \cup P_3)$$

where $P_1 = \left\{v_i : 1 \le i \le \alpha\ where\ \alpha = \frac{r-1}{2}\right\}$ and $P_3 = \left\{v_k : 2\alpha + 2 \le k \le 3\alpha\ where\ \alpha = \frac{r-1}{2}\right\}$, then for every two distinct vertices as t_1 and t_2 in D they are not adjacent, since $t_1 + t_2 \notin Tri(R)$ so D is an independent set. Also for every vertex $v_m \in TP(R)$, where $0 \le m \le 2r - 1$, $m \ne i$ and $m \ne k$, then $d(v_m, t) = 1$ for every $t \in D$. That is; D is an independent dominating set of $|D| = |P_1 \cup P_3| = \alpha + \alpha - 1 = 2\alpha - 1$. Since $\alpha = \frac{r-1}{2}$, so it possesses

$$|D| = 2\left(\frac{r-1}{2}\right) - 1 = r - 2.$$

Now, we show that D is a minimal independent dominating set, assume that $D = P_1$ or $D = P_3$, then there is at least $v_1 \in D$, such that $v_1 + t \notin Tri(R)$, which implies $d(v_1, t) \ge 2$, that is ; D does not independent dominating set. Thus $D = (P_1 \cup P_3)$ is a minimal independent dominating set, with $\gamma_{ind}\left(TP(R)\right) = r - 2$.

Theorem5.8. Let $R \cong Z_{2r}$ where $r > 3$ is a prime integer. Then $M_{dim}\left(TP(R)\right) = r$.

Proof. Let $TP(R)$ be the tripotent graph, $where\ R \cong Z_{2r}$ with $r > 3$ is a prime integer. Consider a resolving set $S \subseteq V(TP(R))$ of the form

$S = \{s_0, s_1, \dots, s_k\}$.

Where $0 \le k \le r - 1$. For any two distinct vertices $t_i, t_j \in V(TP(R))$, their distance representation with respect to S satisfy

$\psi(t_i|S) \ne \psi(t_j|S)$.

Thus, S is a resolving set for $TP(R)$**.** Next to show that S is a minimum resolving set, assume that $1 \le k \le r - 1$. Under this assumption, there exist indices $i \ne j$ such that the corresponding vertices coincide, namely $t_i = t_j$, $i = r - 1, and\ j = 2r - 1$. In this case, we obtain $\psi(t_i|S) \ne \psi(t_j|S)$, which contradicts our assumption. Hence, $S = \{s_0, s_1, \dots, s_k\}$ refer to a minimum resolving set of $TP(R)$, with $|S| = r = M_{dim}\left(TP(R)\right)$.

Conclusion

In conclusion, the tripotent graph of a commutative ring R shows several important structural features, including a well-defined cycle texture and Hamiltonian graph, although it is not Eulerian. Notably, the subgraph formed by nonzero tripotent elements is a line graph, and in local rings, clear connections with the unit graph and the clean graph are observed. A closer look at key graph invariants such as size, diameter, clique number, chromatic number, partiteness, and the independent domination number, especially for reduced rings, highlights the usefulness of the tripotent graph as a tool in algebraic graph theory. In addition, the metric dimension provides further insight and opens new directions for seeking the relationship between ring-theoretic properties and graph textures.

Conflict of interest

None.

References

[1] Anderson, D. F., & Badawi, A. (2008). The total graph of a commutative ring. Journal of Algebra, 320(7), 2706-2719.

[2] Anderson, D. F., & Livingston, P. S. (1999). The zero-divisor graph of a commutative ring. Journal of Algebra, 217(2), 434–447.

[3] Ashraf, N., Maimani, H. R., Pournaki, M. R., & Yassemi, S. (2010). Unit graphs associated with rings. Communications in Algebra, 38(8), 2851–2871.

[4] Atiyah, M. F., & Macdonald, I. G. (2018). Introduction to commutative algebra. CRC Press.

[5] Ali, P. O., and Essa, S. S. (2025). A graph associated with tri-potent elements of a commutative ring. Gulf Journal of Mathematics, 20(1), 405-413.

[6] Chartrand, G., Eroh, L., Johnson, M. A., & Oellermann, O. R. (2000). Resolvability in graphs and the metric dimension of a graph. Discrete Applied Mathematics, 105, 99–113.

[7] Chartrand, G., Jordon, H., Vatter, V., & Zhang, P. (2024). Graphs & Digraphs (7th ed.). Chapman and Hall/CRC.

[8] Dolžan, D. (2016). The Metric Dimension of the Total Graph of a Finite Commutative Ring. Canadian Mathematical Bulletin, 59(4), 748–759.

[9] Goddard, W., & Henning, M. A. (2013). Independent domination in graphs: A survey and recent results. Discrete Mathematics, 313(7), 839–854.

[10] Hummadi, P. A., & Muhammad, A. K. (2010). Smarandache Triple Tripotents in Z_n and in Group Ring Z_2G. International Journal of Algebra, 4(25), 1219–1229.

[11] Khaleel, L. A., Mohammad, H. Q., & Shuker, N. H. (2024). Tripotent divisor graph of a commutative ring. International Journal of Mathematics and Mathematical Sciences, 2024 (Article ID 1954058), 8 pages.

[12] Petrovi ´ c, Z., & Pucanovi ´ c, Z. S. (2017). The clean graph of a commutative ring. Ars Combinatoria, 134, 363‑378.

[13] Razzaghi, S., & Sahebi, S. (2020). A graph with respect to idempotents of a ring. Journal of Algebra and Its Applications, 24(6), 2150105, 11 pages.