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δ − Quasi Triple Operator of Order n

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Abstract

In this study, two new classes of operators on a complex Hilbert space H were presented and referred to as the Quasi-triple operator and δ-quasi-triple operator. Quasi-triple operator is denoted as qt-operator, and the δ-quasi-triple operator is denoted as δ-qt-operator where δ is a linear bounded operator on H. The generalization of the above concepts is Quasi-triple operator of order n and δ-quasi-triple operator of order n. Some results of these two types of operators including sum, product, direct sum and tensor product operations have been discussed. We also presented some results and theorems and supported the discussion with some illustrative examples.

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1. Introduction

In this paper, $\Psi(H)$ is denoted as the algebra of all bounded linear operators on complex Hilbert space H. T is called normal if $TT^* = T^*T$ where T^* is the adjoint operator of T, see [17, 16, 18] and we say that T is unitary if $TT^* = T^*T = I$ [20, 8], also two operators T and S are considered unitarily equivalent when a unitary operator U exists with the property that $\mathrm{T} = \mathrm{USU}^*$, see [13, 19, 15]. In 2015, [12] defined a quasi-normed operator with order n if $T(T^{*n}T^n) = (T^{*n}T^n)T$. In 2016, [5] introduced a new definition of operator called triple operator as $T(TT^*) = (TT^*)T$, and in 2018, [6] defined a new definition of operator called triple operator of order n if $T(T^nT^*) = (T^nT^*)T$, $n \geq 2$, and in 2023, [7] defined a new class of operators called quasi triple operator as $T[(TT^*)T] = [T(TT^*)]T$, and called θ −quasi triple operator as $T[(TT^*)T] = \theta[T(TT^*)]T$ where θ is an operator on H. This study aims to extend these definitions of operator as quasi triple operator with order n if $T[(T^nT^*)T] = [(T^nT^*)T]T$, where n ≥2 and where T is quasi triple operator if $T[(TT^*)T] = [(TT^*)T]T$, also we give a new definition of operators called δ −quasi triple operator with order n as $T[(T^nT^*)T] = \delta[(T^nT^*)T]T$, where δ is a linear bounded operator on H .Also, we introduced the direct sum and the tensor product of n times of quasi triple operator with order n, see [3, 1, 11, 9].

2. Quasi triple operator of order n

Definition 2.1: The operator $\mathfrak{z} \in \Psi(H)$ defined on the Complex Hilbert space H, is designated as Quasi-triple operator denoted by(qt-operator) if and only if $\mathfrak{z}[(\mathfrak{z}\mathfrak{z}^*)\mathfrak{z}] = [(\mathfrak{z}\mathfrak{z}^*)\mathfrak{z}]\mathfrak{z}$.

Definition 2.2*:* The operator $\mathfrak{z} \in \Psi(H)$ on a Hilbert space H is called Quasi-triple operators of orders$n$, denoted by (qt-operator of orders n), iff $\mathfrak{z}[(\mathfrak{z}^n\mathfrak{z}^*)\mathfrak{z}] = [(\mathfrak{z}^n\mathfrak{z}^*)\mathfrak{z}]\mathfrak{z}$, where n ≥*1.* when $n = 1$*,* we get the first definition.

Example 2.3: Let $\mathfrak{z} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$ be an operator on a two-dimensional Hilbert space C^2, it follows that

$\mathfrak{z}[(\mathfrak{z}^2\mathfrak{z}^*)\mathfrak{z}] =$

$$\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}\left[\left(\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}^2 \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}\right)\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}\right]$$

$$= \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}\left[\left(\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}\right)\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}\right]$$

$$= \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ 4 & 4 \end{bmatrix} = [(\mathfrak{z}^n\mathfrak{z}^*)\mathfrak{z}]\mathfrak{z} = \begin{bmatrix} 4 & -4 \\ 4 & 4 \end{bmatrix}$$

Thus $\mathfrak{z}$ is a quasi-triple operator with order 2.

Example 2.4: Let $\mathfrak{z} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & 1 \\ 4 & 2 & 0 \end{bmatrix}$ be an operator of three-dimensional Hilbert space C^3, then

$\mathfrak{z}[(\mathfrak{z}^3\mathfrak{z}^*)\mathfrak{z}] =$

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & 1 \\ 4 & 2 & 0 \end{bmatrix}\left[\left(\begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & 1 \\ 4 & 2 & 0 \end{bmatrix}^3 \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & 1 \\ 4 & 2 & 0 \end{bmatrix}^*\right)\begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & 1 \\ 4 & 2 & 0 \end{bmatrix}\right]$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & 1 \\ 4 & 2 & 0 \end{bmatrix}\left[\left(\begin{bmatrix} 33 & 30 & 6 \\ 57 & 54 & 12 \\ 54 & 48 & 12 \end{bmatrix}\begin{bmatrix} 1 & 3 & 4 \\ 2 & 2 & 2 \\ 0 & 1 & 0 \end{bmatrix}\right)\begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & 1 \\ 4 & 2 & 0 \end{bmatrix}\right]$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & 1 \\ 4 & 2 & 0 \end{bmatrix}\left[\left(\begin{bmatrix} 33 & 30 & 6 \\ 57 & 54 & 12 \\ 54 & 48 & 12 \end{bmatrix}\begin{bmatrix} 1 & 3 & 4 \\ 2 & 2 & 2 \\ 0 & 1 & 0 \end{bmatrix}\right)\begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & 1 \\ 4 & 2 & 0 \end{bmatrix}\right]$$

$$= \begin{bmatrix} 6120 & 4068 & 747 \\ 11040 & 7332 & 1347 \\ 10188 & 6768 & 1242 \end{bmatrix}$$

$$\neq \mathfrak{z}[(\mathfrak{z}^3\mathfrak{z}^*)\mathfrak{z}] = \begin{bmatrix} 4716 & 4842 & 900 \\ 8298 & 8514 & 1584 \\ 2880 & 7884 & 1464 \end{bmatrix}$$

Thus $\mathfrak{z}$ is not a qt-operator with order 3.

Remark (2.5): Every Triple operator of order n is qt-operator of order n but the convers is not true as shown in the next example:

Let $\mathfrak{z} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ be an operator on a two-dimensional Hilbert space C^2, it follows that

$$(\mathfrak{z}^2\mathfrak{z}^*)\mathfrak{z} = \left[\left(\begin{bmatrix} 3 & -5 \\ 5 & 8 \end{bmatrix}^2 \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\right)\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\right] = \begin{bmatrix} 10 & -47 \\ 33 & 85 \end{bmatrix}$$

$$\neq \mathfrak{z}(\mathfrak{z}^2\mathfrak{z}^*) = \begin{bmatrix} 20 & -53 \\ 17 & 75 \end{bmatrix}$$

So $\mathfrak{z}$ is not Triple operator of order 2, but

$$\mathfrak{z}[(\mathfrak{z}^2\mathfrak{z}^*)\mathfrak{z}] = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\left[\left(\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}^2 \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\right)\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\right]$$

$$= \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\left[\left(\begin{bmatrix} 3 & -5 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\right)\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\right]$$

$$= \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 34 & -25 \\ -25 & 89 \end{bmatrix} = \begin{bmatrix} 43 & 39 \\ -109 & 292 \end{bmatrix}$$

$$= [\mathfrak{z}(\mathfrak{z}^2\mathfrak{z}^*)]\mathfrak{z} = \left[\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\left(\begin{bmatrix} 3 & -5 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\right)\right]\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} =$$

$$\left(\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 11 & -12 \\ 2 & 29 \end{bmatrix}\right)\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 24 & 5 \\ -5 & 99 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 43 & 39 \\ -109 & 292 \end{bmatrix}$$

So $\mathfrak{z}$ is qt-operator with order 2.

Proposition 2.6: If η is a qt-operator with order n acting within a Hilbert space H, thus

1- $p\eta$ is qt-operators with orders n, for all complex numbers p.

2- Given that $\mathcal{L}$ is a closed subspace of H, it holds that $\eta|_{\mathcal{L}}$ is qt-operators with orders n.

3- If L is unitary equivalent to η then L is qt-operators with orders n.

Proof:

1- $(p\eta)[((p\eta)^n(p\eta)^*)(p\eta)] =$ $(p\eta)[(p^n\eta^n)(\bar{p}\eta^*))(p\eta)] = pp^n\bar{p}p(\eta[(\eta^n\eta^*)\eta])$, since η is qt-operator of order n, then $pp^n\bar{p}p([(\eta^n\eta^*)\eta]\eta) =$ $[(p^n\eta^n)(\bar{p}\eta^*))(p\eta)](p\eta)$ $=[((p\eta)^n)(p\eta)^*)(p\eta)](p\eta)$ so $(p\eta)$ is qt-operator with order n .

2- $(\eta|_{\mathcal{L}})[((\eta|_{\mathcal{L}})^n(\eta|_{\mathcal{L}})^*)\eta|_{\mathcal{L}}] = (\eta|_{\mathcal{L}})[(\eta^n|_{\mathcal{L}}\eta^*|_{\mathcal{L}})(\eta|_{\mathcal{L}})] =$ $\eta[(\eta^n\eta^*)\eta]|_{\mathcal{L}}$

since η is qt-operators with orders n

$= [(\eta^n\eta^*)\eta)]|_{\mathcal{L}}\,(\eta|_{\mathcal{L}})$

$= [(\eta^n|_{\mathcal{L}}\,)(\eta^*|_{\mathcal{L}})(\eta|_{\mathcal{L}})](\eta|_{\mathcal{L}})$

$= [((\eta|_{\mathcal{L}})^n(\eta|_{\mathcal{L}})^*)\eta|_{\mathcal{L}}]\eta|_{\mathcal{L}}$

Thus $\eta|_{\mathcal{L}}$ is qt-operator with order n.

3- L is related to η by unitary equivalence, therefore, a unitary operator U can be found such that $L = U\eta U^*$ so $L^* = (U\eta U^*)^*$, $L^* = U^{**}\eta^*U^*$ then $L^* = U\eta^*U^*$ $L^n = U\eta^nU^*$, so $L[(L^nL^*)L] =$

$[(U\eta U^*)((U\eta^nU^*)(U\eta^*U^*))(U\eta U^*)]$

$((U\eta U^*)[(U\eta^n\eta^*U^*)(U\eta U^*)] =$

$(U\eta\eta^n\eta^*U^*)(U\eta U^*)$=$U\eta\eta^n\eta^*\eta U^*$

$= U\eta(\eta^n\eta^*\eta)U^*$, since η is qt-operator with order n

$= U[(\eta^n\eta^*)\eta]\eta U^*$ ... ... (1)

On the other hand,

$[(L^nL^*)L]L = [((U\eta^nU^*)(U\eta U^*))(U\eta U^*)](U\eta SU^*)$

$= (U\eta^n\,U^*)(U\eta^*\eta U^*)](U\eta U^*)$

$= (U\eta^n\eta^*\eta\eta U^*)(U\eta U^*)$

$= U\eta^n\eta^*\eta\eta U^*$

$= U([(\eta^n\eta^*)\eta]\eta)U^*$ ... ... (2)

From equations (1) and (2), we get...

$L[(L^nL^*)L] = [(L^nL^*)L]L$, thus L is qt-operator with order n.

Remark 2.7: The sum of two qt-operator with order n is not necessarily qt-operator of order n, as evidenced by the example below.

Example 2.8: The operator $ɳ = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ and $ʒ = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$ operating within a two-dimensional Hilbert space C^2 are qt-operator with order 2.

Since $ɳ[(ɳ^2ɳ^*)ɳ] = \begin{bmatrix} 243 & 0 \\ 0 & 1024 \end{bmatrix} = [(ɳ^nɳ^*)ɳ]ɳ$

and $ʒ\,[(ʒ^nʒ^*)ʒ] = \begin{bmatrix} 4 & -4 \\ 4 & 4 \end{bmatrix} = [(ʒ^nʒ^*)ʒ]ʒ$, but the sum $ɳ + ʒ = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ is not qt-operators with orders 2, since

$(ɳ+ʒ)[((ɳ+ʒ)^2(ɳ+ʒ)^*)(ɳ+ʒ)] = \begin{bmatrix} -13 & -179 \\ 109 & 208 \end{bmatrix} \neq$

$[((ɳ+ʒ)^2(ɳ+ʒ)^*)(ɳ+ʒ)](ɳ+ʒ) = \begin{bmatrix} -27 & -151 \\ 151 & 222 \end{bmatrix}$

Thus $(ɳ+ʒ)$ is not qt-operators with orders 2.

Theorem 2.9: If $ɳ, ʒ$ represent qt-operator with order n for which $ɳ, ʒ$ are commuting normal-operator, such that $ɳ^nʒ^n = 0\ for\ n \geq 2$, then $(ɳ+ʒ)$ is qt-operatorwith order n.

Proof: Since $ɳ, ʒ$ are commuting normal operator

$ɳʒ = ʒɳ$ ... (1)

and by (Putnam -Fuglede theorem) [10, 2], we have

$ɳʒ^* = ʒ^*ɳ$ ... (2), and

$ʒɳ^* = ɳ^*ʒ$ ... (3)

Now, $(ɳ+ʒ)[((ɳ+ʒ)^n(ɳ+ʒ)^*)(ɳ+ʒ)] = (ɳ+ʒ)[((ɳ^n+ʒ^n)(ɳ^*+ʒ^*))(ɳ+ʒ)]$

$= (ɳ+ʒ)[(ɳ^nɳ^* + ɳ^nʒ^* + ʒ^nɳ^* + ʒ^nʒ^*)(ɳ+ʒ)]$

$= (ɳ+ʒ)[(ɳ^nɳ^*ɳ + ɳ^nʒ^*ɳ + ʒ^nɳ^*ɳ + ʒ^nʒ^*ɳ + ɳ^nɳ^*ʒ + ɳ^nʒ^*ʒ + ʒ^nɳ^*ʒ + ʒ^nʒ^*ʒ)]$

$= ɳɳ^nɳ^*ɳ + ɳɳ^nʒ^*ɳ + ɳʒ^nɳ^*ɳ + ɳʒ^nT^*ɳ + ɳɳ^nɳ^*ʒ + ɳɳ^nʒ^*ʒ + ɳʒ^nɳ^*ʒ + ɳʒ^nʒ^*ʒ + ʒɳ^nɳ^*ɳ + ʒɳ^nʒ^*ɳ + ʒʒ^nɳ^*ɳ + ʒʒ^nʒ^*ɳ + ʒɳ^nɳ^*ʒ + ʒɳ^nʒ^*ʒ + ʒʒ^nɳ^*ʒ + ʒʒ^nʒ^*ʒ$

From the equations (1), (2) and (3), and since $ɳ, ʒ$ are quasi-triple operator with order n, we have

$= ɳ^nɳ^*ɳɳ + ɳ^nʒ^*ɳɳ + ʒ^nɳ^*ɳɳ + ʒ^nʒ^*ɳɳ + ɳ^nɳ^*ɳʒ + ɳ^nʒ^*ɳʒ + ʒ^nɳ^*ɳʒ + ʒ^nʒ^*ɳʒ + ɳ^nɳ^*ʒɳ + ɳ^nʒ^*ʒɳ + ʒ^nɳ^*ʒɳ + ʒ^nʒ^*ʒɳ + ɳ^nɳ^*ʒʒ + ɳ^nʒ^*ʒʒ + ʒ^nɳ^*ʒʒ + ʒ^nʒ^*ʒʒ$

$= (ɳ^nɳ^*ɳ + ɳ^nʒ^*ɳ + ʒ^nɳ^*ɳ + ʒ^nʒ^*ɳ + ɳ^nɳ^*ʒ + ɳ^nʒ^*ʒ + ʒ^nɳ^*ʒ + ʒ^nʒ^*ʒ)ɳ + (ɳ^nɳ^*ɳ + ɳ^nʒ^*ɳ + ʒ^nɳ^*ɳ + ʒ^nʒ^*ɳ + ɳ^nɳ^*ʒ + ɳ^nʒ^*ʒ + ʒ^nɳ^*ʒ + ʒ^nʒ^*ʒ)ʒ$

$= [(ɳ^nɳ^*ɳ + ɳ^nʒ^*ɳ + ʒ^nɳ^*ɳ + ʒ^nʒ^*ɳ + ɳ^nɳ^*ʒ + ɳ^nʒ^*ʒ + ʒ^nɳ^*ʒ + ʒ^nʒ^*ʒ)](ɳ+ʒ)$

$= [(ɳ^nɳ^* + ɳ^nʒ^* + ʒ^nɳ^* + ʒ^nʒ^*)ɳ + (ɳ^nɳ^* + ɳ^nʒ^* + ʒ^nɳ^* + ʒ^nʒ^*)ʒ](ɳ+ʒ)$

$= [(ɳ^nɳ^* + ɳ^nʒ^* + ʒ^nɳ^* + ʒ^nʒ^*)(ɳ+ʒ)](ɳ+ʒ)$

$= [(ɳ^n+ʒ^n)(ɳ^*+ʒ^*)(ɳ+ʒ)](ɳ+ʒ)$

$= [(ɳ+ʒ)^n(ɳ+ʒ)^*(ɳ+ʒ)](ɳ+ʒ)$

So $(ɳ+ʒ)$ is qt-operator with order n

Remark 2.10: The product of two qt-operator doesn't need to be qt-operator with order n, as seen in the example below

Example 2.11: Let $ʒ = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ and $ɳ = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$ $ɳ, ʒ$ are qt-operator of order n

$[(ɳ^nɳ^*)ɳ]ɳ + [(ʒ^nʒ^*)ʒ]ʒ = [((ɳʒ)^n(ɳʒ)^*)(ɳʒ)](ɳʒ)$

Since $[((ɳʒ)^2(ɳʒ)^*)(ɳʒ)](ɳʒ) = \begin{bmatrix} 2850 & -3368 \\ 1518 & 1000 \end{bmatrix} \neq \begin{bmatrix} 1674 & -3200 \\ 1350 & 2176 \end{bmatrix} = (ɳʒ)[((ɳʒ)^n(ɳʒ)^*)(ɳʒ)]$.

Then, $(ɳʒ)$ is not qt-operator of order 2.

Theorem 2.12: If ʒis defined as a qt-operator with order n while ɳ is an isometry operator for which

1- $ɳʒ = ʒɳ$,

2- $ɳ^nʒ = ʒɳ^n$

3- $ʒ^nɳ = ɳʒ^n$ then $(ɳʒ)$ is classified as a qt-operator of the nth order.

Proof:

Given that $ɳʒ = ʒɳ$, thus $ɳ^*ʒ = ʒɳ^*$ and $ʒ^*ɳ = ɳʒ^*$ by (Putnam -Fuglede theorem),

and because ɳ is an isometry operator, it follows that $ɳ^*ɳ = I$ from [4, 14]

$(ɳʒ)[((ɳʒ)^n(ɳʒ)^*)(ɳʒ)]$

$= (ɳʒ)[(ɳ^nʒ^n)(ʒ^*ɳ^*)(ɳʒ))]$

$= (ɳʒ)[(ɳ^nʒ^nʒ^*ɳ^*)(ɳʒ))]$

$= (ɳʒ)(ɳ^nʒ^nʒ^*ʒ)$

$= ɳʒɳ^nʒ^nʒ^*ʒ$

$= ɳɳ^n[ʒ(ʒ^nʒ^*)ʒ]$, ʒ is qt-operators with orders n

$= ɳɳ^n[(ʒ^nʒ^*)ʒ]ʒ$

$= (ɳɳ^nʒ^nʒ^*ʒʒ)ɳ^*ɳ$ $ɳ^*ɳ = I$

$= (ɳ^nɳʒ^nʒ^*ʒʒ)ɳ^*ɳ$ $ɳʒ^n = ʒ^nɳ$

$= (ɳ^nʒ^nɳʒ^*ʒʒ)ɳ^*ɳ$ $ɳʒ^* = ʒ^*ɳ$

$= (ɳ^nʒ^nʒ^*ɳʒʒ)ɳ^*ɳ$

$= ɳ^nʒ^nʒ^*ɳʒɳ^*ʒɳ$

$= (ɳʒ)^nʒ^*ɳɳ^*ʒʒɳ$ $ɳɳ^* = I$

$= (ɳʒ)^nʒ^*ɳ^*ɳʒʒɳ$

$= (ɳʒ)^nʒ^*ɳ^*ɳʒɳʒ$

$= [((ɳʒ)^n(ɳʒ)^*)(ɳʒ)](ɳʒ)$

ɳʒis qt-operators with orders n.

Theorem 2.13: If $ʒ, ɳ$ are operators, and ʒ and ɳ are unitary-equivalent and ɳ is normal, then ʒ is qt-operator.

Proof: Given that ʒ is unitary-equivalent to ɳ, it follows that $ʒ = U^*ɳU$

To prove that ʒ is qt-operator with order n

$ʒ[(ʒʒ^*)ʒ] = (U^*ɳU)[(U^*ɳU)(U^*ɳ^*U))(U^*ɳU)]$

$= (U^*ɳU)[(U^*ɳɳ^*U))(U^*ɳU)]$

$= U^*(ɳ[(ɳɳ^*)ɳ]U$, since ɳ is normal

therefore, $U^*(ɳ[(ɳɳ^*)ɳ]U = U^*(ɳ[(ɳ^*ɳ)ɳ]U$

$= U^*[((ɳɳ^*)ɳ)ɳ])U$

$= U^*[((ɳ(UU^*)ɳ^*(UU^*))ɳ(UU^*))ɳ])U$

$= [((U^*ɳU)(U^*ɳ^*U))(U^*ɳU))](U^*ɳU)$

$= [(ʒʒ^*)ʒ]ʒ$.

Therefore, ʒ is qt-operator with order n

Theorem 2.14: If $\mathfrak{z}_1, \mathfrak{z}_2, \ldots, \mathfrak{z}_l$ are qt-operator of order n, then the direct sum $(\mathfrak{z}_1 \oplus \mathfrak{z}_2 \ldots \oplus \mathfrak{z}_l)$ is qt-operator of order n...

Proof:

$(\mathfrak{z}_1 \oplus \mathfrak{z}_2 \ldots \oplus \mathfrak{z}_l)\big[\big((\mathfrak{z}_1 \oplus \mathfrak{z}_2 \ldots \oplus \mathfrak{z}_l)^n (\mathfrak{z}_1 \oplus \mathfrak{z}_2 \ldots \oplus \mathfrak{z}_l)^*\big)\big) \oplus (\mathfrak{z}_1 \oplus \mathfrak{z}_2 \ldots \oplus \mathfrak{z}_l)\big]$
$= (\mathfrak{z}_1 \oplus \mathfrak{z}_2 \ldots \oplus \mathfrak{z}_l)[((\mathfrak{z}_1{}^n \oplus \mathfrak{z}_2{}^n \ldots . \oplus \mathfrak{z}_l{}^n)(\mathfrak{z}_1{}^* \oplus \mathfrak{z}_2{}^* . \ldots \oplus \mathfrak{z}_l{}^*))(\mathfrak{z}_1 \oplus \mathfrak{z}_2 \ldots \oplus \mathfrak{z}_l)]$
$= (\mathfrak{z}_1 \oplus \mathfrak{z}_2 \ldots \oplus \mathfrak{z}_l)[(\mathfrak{z}_1{}^n \mathfrak{z}_1{}^*)\mathfrak{z}_1 \oplus (\mathfrak{z}_2{}^n \mathfrak{z}_2{}^*)\mathfrak{z}_2 \ldots . \oplus (\mathfrak{z}_l{}^n \mathfrak{z}_l{}^*)\mathfrak{z}_l]$
$= \mathfrak{z}_1[(\mathfrak{z}_1{}^n \mathfrak{z}_1{}^*)\, \mathfrak{z}_1] \oplus \mathfrak{z}_2[(\mathfrak{z}_2{}^n \mathfrak{z}_2{}^*)\mathfrak{z}_2] \oplus \ldots . \oplus \mathfrak{z}_l[(\, \mathfrak{z}_l{}^n \mathfrak{z}_l{}^*)\mathfrak{z}_l]$
Since $\mathfrak{z}_1, , \mathfrak{z}_2, \ldots, \mathfrak{z}_l$ are qt-operator with orders n, then
$= [(\mathfrak{z}_1{}^n \mathfrak{z}_1{}^*)\mathfrak{z}_1]\mathfrak{z}_1 \oplus [(\mathfrak{z}_2{}^n \mathfrak{z}_2{}^*)\mathfrak{z}_2]\mathfrak{z}_2 \ldots \oplus [(\mathfrak{z}_l{}^n \mathfrak{z}_l{}^*)\mathfrak{z}_l]\mathfrak{z}_l$
$= [(\mathfrak{z}_1{}^n \mathfrak{z}_1{}^*)\mathfrak{z}_1 \oplus (\mathfrak{z}_2{}^n \mathfrak{z}_2{}^*)\mathfrak{z}_2 \ldots \oplus (\mathfrak{z}_l{}^n \mathfrak{z}_l{}^*)\mathfrak{z}_l](\mathfrak{z}_1 \oplus \mathfrak{z}_2 \ldots \oplus \mathfrak{z}_l)$
$= \big[\big((\mathfrak{z}_1{}^n \mathfrak{z}_1{}^*) \oplus (\mathfrak{z}_2{}^n \mathfrak{z}_2{}^*) \ldots \oplus (\mathfrak{z}_l{}^n \mathfrak{z}_l{}^*)\big)(\mathfrak{z}_1 \oplus \mathfrak{z}_2 \ldots \oplus \mathfrak{z}_l)\big] (\mathfrak{z}_1 \oplus \mathfrak{z}_2 \ldots \oplus \mathfrak{z}_l)$
$= \big[\big((\mathfrak{z}_1{}^n \oplus \mathfrak{z}_2{}^n \ldots \oplus \mathfrak{z}_l{}^n)(\mathfrak{z}_1{}^* \oplus \mathfrak{z}_2{}^* \ldots \oplus \mathfrak{z}_l{}^*)\big) (\mathfrak{z}_1 \oplus \mathfrak{z}_2 \ldots \oplus \mathfrak{z}_l)\big](\mathfrak{z}_1 \oplus \mathfrak{z}_2 \ldots \oplus \mathfrak{z}_l)$
$= [((\mathfrak{z}_1 \oplus \mathfrak{z}_2 \ldots \oplus \mathfrak{z}_l)^n (\mathfrak{z}_1 \oplus \mathfrak{z}_2 \ldots \oplus \mathfrak{z}_l)^*)(\mathfrak{z}_1 \oplus \mathfrak{z}_2 \ldots \oplus \mathfrak{z}_l)] (\mathfrak{z}_1 \oplus \mathfrak{z}_2 \ldots \oplus \mathfrak{z}_l)$
Thus $\mathfrak{z}_1 \oplus \mathfrak{z}_2 \ldots \oplus \mathfrak{z}_l$ is qt-operator with order n

Theorem 2.15: If$\mathfrak{z}_1, \mathfrak{z}_2, \ldots \mathfrak{z}_l$ are qt-operatorwith order n, then the tensor product $(\mathfrak{z}_1 \otimes \ldots . \otimes \mathfrak{z}_l)$ is qt-operator with order n.

Proof:

$(\mathfrak{z}_1 \otimes \ldots . \otimes \mathfrak{z}_l)[[(\mathfrak{z}_1 \otimes \ldots . \otimes \mathfrak{z}_l)^n (\mathfrak{z}_1 \otimes \ldots . \otimes \mathfrak{z}_l)^*] (\mathfrak{z}_1 \otimes \ldots . \otimes \mathfrak{z}_l)]$
$= (\mathfrak{z}_1 \otimes \ldots . \otimes \mathfrak{z}_l)[[(\mathfrak{z}_1{}^n \otimes . \ldots \otimes \mathfrak{z}_l{}^n)(\mathfrak{z}_1{}^* \otimes . \ldots \otimes \mathfrak{z}_l{}^*)] (\mathfrak{z}_1 \otimes \ldots . \otimes \mathfrak{z}_l)]$
$= (\mathfrak{z}_1 \otimes \ldots . \otimes \mathfrak{z}_l)[[\mathfrak{z}_1{}^n \mathfrak{z}_1{}^* \otimes \ldots . \otimes \mathfrak{z}_l{}^n \mathfrak{z}_l{}^*](\mathfrak{z}_1 \otimes \ldots . \otimes \mathfrak{z}_l)]$
$= (\mathfrak{z}_1 \otimes \ldots . \otimes \mathfrak{z}_l)[(\mathfrak{z}_1{}^n \mathfrak{z}_1{}^*)\mathfrak{z}_1 \otimes \ldots \otimes (\mathfrak{z}_l{}^n \mathfrak{z}_l{}^*)\mathfrak{z}_l]$
$= \mathfrak{z}_1[(\mathfrak{z}_1{}^n \mathfrak{z}_1{}^*)\mathfrak{z}_1] \otimes \ldots \otimes \mathfrak{z}_l[(\mathfrak{z}_l{}^n \mathfrak{z}_l{}^*)\mathfrak{z}_l]$
Since $\mathfrak{z}_1, \mathfrak{z}_2, \ldots, \mathfrak{z}_l$ are qt-operator with order n, then
$[\mathfrak{z}_1[(\mathfrak{z}_1{}^n \mathfrak{z}_1{}^*)\mathfrak{z}_1] \otimes \ldots \otimes \mathfrak{z}_l[(\mathfrak{z}_l{}^n \mathfrak{z}_l{}^*)\mathfrak{z}_l]$
$= [(\mathfrak{z}_1{}^n \mathfrak{z}_1{}^*)\mathfrak{z}_1]\mathfrak{z}_1 \otimes \ldots \otimes [(\mathfrak{z}_l{}^n \mathfrak{z}_l{}^*)\mathfrak{z}_l]\mathfrak{z}_l$
$= [(\mathfrak{z}_1{}^n \mathfrak{z}_1{}^*)\mathfrak{z}_1 \otimes \ldots \otimes (\mathfrak{z}_l{}^n \mathfrak{z}_l{}^*)\mathfrak{z}_l](\mathfrak{z}_1 \otimes \ldots . \otimes \mathfrak{z}_l)$
$= [((\mathfrak{z}_1{}^n \otimes \ldots \otimes \mathfrak{z}_l{}^n)(\mathfrak{z}_1{}^* \otimes \ldots \otimes \mathfrak{z}_l{}^*))(\mathfrak{z}_1 \otimes \ldots . \otimes \mathfrak{z}_l)](\mathfrak{z}_1 \otimes \ldots . \otimes \mathfrak{z}_l)$
$= [((\mathfrak{z}_1 \otimes \ldots . \otimes \mathfrak{z}_l)^n (\mathfrak{z}_1 \otimes \ldots . \otimes \mathfrak{z}_l)^*)(\mathfrak{z}_1 \otimes \ldots . \otimes \mathfrak{z}_l)] (\mathfrak{z}_1 \otimes \ldots . \otimes \mathfrak{z}_l)$
Therefore,$\mathfrak{z}_1 \otimes \ldots . \otimes \mathfrak{z}_l$is qt-operator with order n

3. δ-quasi triple operator with order n

Definition 3.1: The operator $\mathfrak{z}$ within the Hilbert space H is known as δ-quasi triple operator with order n denoted by (δ-qt-operator with order n) if $\mathfrak{z}[(\mathfrak{z}^n \mathfrak{z}^*)\mathfrak{z}] = \delta[(\mathfrak{z}^n \mathfrak{z}^*)\mathfrak{z}]\mathfrak{z}$, where δ is a linear bounded operator on H.

Remark 3.2: Every qt-operator with order n is δ-qt-operator with order n when $\delta = I$ but the opposite is not true, as an example:

Example 3.3: Let $\mathfrak{z} = \begin{bmatrix}1 & 2\\0 & 3\end{bmatrix}$ be an operator on C^2, then

$\mathfrak{z}[(\mathfrak{z}^2 \mathfrak{z}^*)\mathfrak{z}] = \begin{bmatrix}1 & 2\\0 & 3\end{bmatrix}\left[\left(\begin{bmatrix}1 & 2\\0 & 3\end{bmatrix}^2 \begin{bmatrix}1 & 0\\2 & 3\end{bmatrix}\right)\begin{bmatrix}1 & 2\\0 & 3\end{bmatrix}\right] = \begin{bmatrix}53 & 340\\54 & 351\end{bmatrix}$

$[(\mathfrak{z}^n \mathfrak{z}^*)\mathfrak{z}]\mathfrak{z} = \left[\left(\begin{bmatrix}1 & 2\\0 & 3\end{bmatrix}^2 \begin{bmatrix}1 & 0\\2 & 3\end{bmatrix}\right)\begin{bmatrix}1 & 2\\0 & 3\end{bmatrix}\right]\begin{bmatrix}1 & 2\\0 & 3\end{bmatrix} = \begin{bmatrix}17 & 352\\18 & 387\end{bmatrix}$, so $\mathfrak{z}$ is not qt-operator with order 2.

But if we take $\delta = \begin{bmatrix}53/17 & 340/352\\54/18 & 351/387\end{bmatrix}$, this will be

$[(\mathfrak{z}^n \mathfrak{z}^*)\mathfrak{z}]\mathfrak{z} = \begin{bmatrix}1 & 2\\0 & 3\end{bmatrix}\left[\left(\begin{bmatrix}1 & 2\\0 & 3\end{bmatrix}^2 \begin{bmatrix}1 & 0\\2 & 3\end{bmatrix}\right)\begin{bmatrix}1 & 2\\0 & 3\end{bmatrix}\right] = \begin{bmatrix}53 & 340\\54 & 351\end{bmatrix}$

On the other hand, $\delta[(\mathfrak{z}^n \mathfrak{z}^*)\mathfrak{z}]\mathfrak{z} =$
$\begin{bmatrix}53/17 & 340/352\\54/18 & 351/387\end{bmatrix}\left[\left(\begin{bmatrix}1 & 2\\0 & 3\end{bmatrix}^2 \begin{bmatrix}1 & 0\\2 & 3\end{bmatrix}\right)\begin{bmatrix}1 & 2\\0 & 3\end{bmatrix}\right]\begin{bmatrix}1 & 2\\0 & 3\end{bmatrix}$
$= \begin{bmatrix}53 & 340\\54 & 351\end{bmatrix}$, thus $[(\mathfrak{z}^n \mathfrak{z}^*)\mathfrak{z}]\mathfrak{z} = \delta[(\mathfrak{z}^n \mathfrak{z}^*)\mathfrak{z}]\mathfrak{z}$, so $\mathfrak{z}$ is δ-qt-operator with order 2.

Theorem 3.4 If $\mathfrak{z}$ and ηbe twoδ $-$qt-operatorswith order n on Hilbert space H such that $\eta^n \mathfrak{z}^* = \mathfrak{z}^n \eta^* = \mathfrak{z}^* \eta = \eta^* \mathfrak{z} = 0$ then $\eta + \mathfrak{z}$is also δ $-$qt- operator with order n.

Proof: Let $\mathfrak{z}$ and η be δ-qt-operators with order n, let
$(\eta + \mathfrak{z})[((\eta + \mathfrak{z})^n (\eta + \mathfrak{z})^*)(\eta + \mathfrak{z})] = (\eta + \mathfrak{z})[(\eta^n + \mathfrak{z}^n)(\eta^* + \mathfrak{z}^*))(\eta + \mathfrak{z})]$
$= (\eta + \mathfrak{z})[(\eta^n \eta^* + \eta^n \mathfrak{z}^* + \mathfrak{z}^n \eta^* + \mathfrak{z}^n \mathfrak{z}^*)(\eta + \mathfrak{z})]$
$= (\eta + \mathfrak{z})[(\eta^n \eta^* \eta + \eta^n \mathfrak{z}^* \eta + \mathfrak{z}^n \eta^* \eta + \mathfrak{z}^n \mathfrak{z}^* \eta + \eta^n \eta^* \mathfrak{z} + \eta^n \mathfrak{z}^* \mathfrak{z} + \mathfrak{z}^n \eta^* \mathfrak{z} + \mathfrak{z}^n \mathfrak{z}^* \mathfrak{z})]$
$= \eta\eta^n \eta^* \eta + \eta\eta^n \mathfrak{z}^* \eta + \eta\mathfrak{z}^n \eta^* \eta + \eta\mathfrak{z}^n \mathfrak{z}^* \eta + \eta\eta^n \eta^* \mathfrak{z} + \eta\eta^n \mathfrak{z}^* \mathfrak{z} + \eta\mathfrak{z}^n \eta^* \mathfrak{z} + \eta\mathfrak{z}^n \mathfrak{z}^* \mathfrak{z} + \mathfrak{z}\eta^n \eta^* \eta + \mathfrak{z}\eta^n \mathfrak{z}^* \eta + \mathfrak{z}\mathfrak{z}^n \eta^* \eta + \mathfrak{z}\mathfrak{z}^n \mathfrak{z}^* \eta + \mathfrak{z}\eta^n \eta^* \mathfrak{z} + \mathfrak{z}\eta^n \mathfrak{z}^* \mathfrak{z} + \mathfrak{z}\mathfrak{z}^n \eta^* \mathfrak{z} + \mathfrak{z}\mathfrak{z}^n \mathfrak{z}^* \mathfrak{z}$,
since $\eta^n \mathfrak{z}^* = \mathfrak{z}^n \eta^* = \eta^* \mathfrak{z} = \eta^* \mathfrak{z} = \mathfrak{z}^n \eta^* = 0$
Then, $(\eta + \mathfrak{z})[((\eta + \mathfrak{z})^n (\eta + \mathfrak{z})^*)(\eta + \mathfrak{z})] = \eta\eta^n \eta^* \eta + \mathfrak{z}\mathfrak{z}^n \mathfrak{z}^* \mathfrak{z} = \eta[(\eta^n \eta^*)\eta] + \mathfrak{z}[(\mathfrak{z}^n \mathfrak{z}^*)\mathfrak{z}]$
Since η and $\mathfrak{z}$ are δ-qt-operators with order n, thus
$= \eta[(\eta^n \eta^*)\eta] + \mathfrak{z}[(\mathfrak{z}^n \mathfrak{z}^*)\mathfrak{z}] = \delta[(\eta^n \eta^*)\eta]\eta + \delta[(\mathfrak{z}^n \mathfrak{z}^*)\mathfrak{z}]\mathfrak{z}$
$= \delta\big[[(\eta^n \eta^*)\eta]\eta + [(\mathfrak{z}^n \mathfrak{z}^*)\mathfrak{z}]\mathfrak{z}\big] = \delta\left[[((\eta^n + \mathfrak{z}^n)(\eta^* + \mathfrak{z}^*))(\eta + \mathfrak{z})](\eta + \mathfrak{z})\right]$
$= \delta\big[[((\eta + \mathfrak{z})^n (\eta + \mathfrak{z})^*)(\eta + \mathfrak{z})](\eta + \mathfrak{z})\big]$
Therefore, $(\eta + \mathfrak{z})$ is δ-qt-operator with order n

Remark (3.5): In Theorem (3.4), if the condition $\eta^n \mathfrak{z}^* = \mathfrak{z}^n \eta^* = \mathfrak{z}^* \eta = \eta^* \mathfrak{z} = 0$ is achieved, then η and $\mathfrak{z}$ doesn't need to be zero as in the following example...

Let $\mathfrak{z} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\eta = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $\eta, \mathfrak{z}$ be δ −qt-operators of order 2, when $\delta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\eta^n \mathfrak{z}^* = \mathfrak{z}^n \eta^* = \mathfrak{z}^* \eta = \eta^* \mathfrak{z} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$(\eta + \mathfrak{z})[((\eta + \mathfrak{z})^2 (\eta + \mathfrak{z})^*)(\eta + \mathfrak{z})] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \delta[((\eta + \mathfrak{z})^2 (\eta + \mathfrak{z})^*)(\eta + \mathfrak{z})](\eta + \mathfrak{z})$

So $(\eta + \mathfrak{z})$ is δ −qt-operator of order 2

Theorem 3.6: If $\mathfrak{z}_1, \mathfrak{z}_2, \dots, \mathfrak{z}_l$ are δ −qt-operators with order n, then the direct sum $(\mathfrak{z}_1 \oplus \mathfrak{z}_2 \dots \oplus \mathfrak{z}_l)$ is δ −qt-operator of order n.

Proof:

$(\mathfrak{z}_1 \oplus \mathfrak{z}_2 \dots \oplus \mathfrak{z}_l)$

$[[(\mathfrak{z}_1 \oplus \mathfrak{z}_2 \dots \oplus \mathfrak{z}_l)^n (\mathfrak{z}_1 \oplus \mathfrak{z}_2 \dots \oplus \mathfrak{z}_l)^*)](\mathfrak{z}_1 \oplus \mathfrak{z}_2 \dots \oplus \mathfrak{z}_l)]$

$= (\mathfrak{z}_1 \oplus \mathfrak{z}_2 \dots \oplus \mathfrak{z}_l)$

$[[(\mathfrak{z}_1 \oplus \mathfrak{z}_2 \dots \oplus \mathfrak{z}_l)^n (\mathfrak{z}_1 \oplus \mathfrak{z}_2 \dots \oplus \mathfrak{z}_l)^*)](\mathfrak{z}_1 \oplus \mathfrak{z}_2 \dots \oplus \mathfrak{z}_l)]$

$= (\mathfrak{z}_1 \oplus \mathfrak{z}_2 \dots \oplus \mathfrak{z}_l)[((\mathfrak{z}_1{}^n \oplus \mathfrak{z}_2{}^n \dots \oplus \mathfrak{z}_l{}^n)(\mathfrak{z}_1{}^* \oplus \mathfrak{z}_2{}^* \dots \oplus \mathfrak{z}_l{}^*))\mathfrak{z}_1 \oplus \mathfrak{z}_2 \dots \oplus \mathfrak{z}_l]$

$= \mathfrak{z}_1 \oplus \mathfrak{z}_2 \dots \oplus \mathfrak{z}_l[(\mathfrak{z}_1{}^n \mathfrak{z}_1{}^*)\mathfrak{z}_1 \oplus (\mathfrak{z}_2{}^n \mathfrak{z}_2{}^*)\mathfrak{z}_2 \dots \oplus (\mathfrak{z}_l{}^n \mathfrak{z}_l{}^*)\mathfrak{z}_l]$

$= \mathfrak{z}_1[(\mathfrak{z}_1{}^n \mathfrak{z}_1{}^*)\mathfrak{z}_1] \oplus \mathfrak{z}_2[(\mathfrak{z}_2{}^n \mathfrak{z}_2{}^*)\mathfrak{z}_2] \dots \oplus \mathfrak{z}_l[(\mathfrak{z}_l{}^n \mathfrak{z}_l{}^*)\mathfrak{z}_l]$

Since $\mathfrak{z}_1, \mathfrak{z}_2, \dots, \mathfrak{z}_l$ are δ −qt-operators with order n, then

$= \delta[(\mathfrak{z}_1{}^n \mathfrak{z}_1{}^*)\mathfrak{z}_1]\mathfrak{z}_1 \oplus \delta[(\mathfrak{z}_2{}^n \mathfrak{z}_2{}^*)\mathfrak{z}_2]\mathfrak{z}_2 \dots \oplus \delta[(\mathfrak{z}_l{}^n \mathfrak{z}_l{}^*)\mathfrak{z}_l]\mathfrak{z}_l$

$= \delta[[(\mathfrak{z}_1{}^n \mathfrak{z}_1{}^*)\mathfrak{z}_1]\mathfrak{z}_1 \oplus [(\mathfrak{z}_2{}^n \mathfrak{z}_2{}^*)\mathfrak{z}_2]\mathfrak{z}_2 \dots \oplus [(\mathfrak{z}_l{}^n \mathfrak{z}_l{}^*)\mathfrak{z}_l]\mathfrak{z}_l]$

$= \delta[[(\mathfrak{z}_1{}^n \mathfrak{z}_1{}^*)\mathfrak{z}_1 \oplus (\mathfrak{z}_2{}^n \mathfrak{z}_2{}^*)\mathfrak{z}_2 \dots \oplus (\mathfrak{z}_l{}^n \mathfrak{z}_l{}^*)\mathfrak{z}_l](\mathfrak{z}_1 \oplus \mathfrak{z}_2 \dots \oplus \mathfrak{z}_l)]$

$= \delta[[((\mathfrak{z}_1{}^n \mathfrak{z}_1{}^*) \oplus (\mathfrak{z}_2{}^n \mathfrak{z}_2{}^*) \dots \oplus (\mathfrak{z}_l{}^n \mathfrak{z}_l{}^*))(\mathfrak{z}_1 \oplus \mathfrak{z}_2 \dots \oplus \mathfrak{z}_l)](\mathfrak{z}_1 \oplus \mathfrak{z}_2 \dots \oplus \mathfrak{z}_l)]$

$= \delta[[((\mathfrak{z}_1{}^n \oplus \mathfrak{z}_2{}^n \dots \oplus \mathfrak{z}_l{}^n)(\mathfrak{z}_1{}^* \oplus \mathfrak{z}_2{}^* \dots \oplus \mathfrak{z}_l{}^*))(\mathfrak{z}_1 \oplus \mathfrak{z}_2 \dots \oplus \mathfrak{z}_l)](\mathfrak{z}_1 \oplus \mathfrak{z}_2 \dots \oplus \mathfrak{z}_l)]$

$= \delta[[((\mathfrak{z}_1 \oplus \mathfrak{z}_2 \dots \oplus \mathfrak{z}_l)^n (\mathfrak{z}_1 \oplus \mathfrak{z}_2 \dots \oplus \mathfrak{z}_l)^*)(\mathfrak{z}_1 \oplus \mathfrak{z}_2 \dots \oplus \mathfrak{z}_l)]$

$(\mathfrak{z}_1 \oplus \mathfrak{z}_2 \dots \oplus \mathfrak{z}_l)]$

Hence $\mathfrak{z}_1 \oplus \mathfrak{z}_2 \dots \oplus \mathfrak{z}_l$ is δ-qt-operator with order n.

Theorem 3.7: If $\mathfrak{z}_1, \mathfrak{z}_2, \dots, \mathfrak{z}_l$ are δ–qt-operators with order n, then the tensor product $(\mathfrak{z}_1 \otimes \dots \otimes \mathfrak{z}_l)$ is δ -qt-operator of order n.

Proof:

$(\mathfrak{z}_1 \otimes \dots \otimes \mathfrak{z}_l)[[(\mathfrak{z}_1 \otimes \dots \otimes \mathfrak{z}_l)^n (\mathfrak{z}_1 \otimes \dots \otimes \mathfrak{z}_l)^*](\mathfrak{z}_1 \otimes \dots \otimes \mathfrak{z}_l)]$

$= (\mathfrak{z}_1 \otimes \dots \otimes \mathfrak{z}_l)[[(\mathfrak{z}_1{}^n \otimes \dots . \otimes \mathfrak{z}_l{}^n)(\mathfrak{z}_1{}^* \otimes \dots . \otimes \mathfrak{z}_l{}^*)](\mathfrak{z}_1 \otimes \dots . \otimes \mathfrak{z}_l)]$

$= (\mathfrak{z}_1 \otimes \dots \otimes \mathfrak{z}_l)[[\mathfrak{z}_1{}^n \mathfrak{z}_1{}^* \otimes \dots . \otimes \mathfrak{z}_l{}^n \mathfrak{z}_l{}^*](\mathfrak{z}_1 \otimes \dots . \otimes \mathfrak{z}_l)]$

$= (\mathfrak{z}_1 \otimes \dots \otimes \mathfrak{z}_l)[(\mathfrak{z}_1{}^n \mathfrak{z}_1{}^*)\mathfrak{z}_1 \dots \otimes (\mathfrak{z}_l{}^n \mathfrak{z}_l{}^*)\mathfrak{z}_l]$

$= \mathfrak{z}_1[(\mathfrak{z}_1{}^n \mathfrak{z}_1{}^*)\mathfrak{z}_1] \otimes \dots \otimes \mathfrak{z}_l[(\mathfrak{z}_l{}^n \mathfrak{z}_l{}^*)\mathfrak{z}_l]$

Since $\mathfrak{z}_1, \mathfrak{z}_2, \dots, \mathfrak{z}_l$ are δ -qt-operators with order n, then

$[\mathfrak{z}_1[(\mathfrak{z}_1{}^n \mathfrak{z}_1{}^*)\mathfrak{z}_1] \otimes \dots \otimes \mathfrak{z}_l[(\mathfrak{z}_l{}^n \mathfrak{z}_l{}^*)\mathfrak{z}_l]$

$= \delta[(\mathfrak{z}_1{}^n \mathfrak{z}_1{}^*)\mathfrak{z}_1]\mathfrak{z}_1 \otimes \dots \otimes \delta[(\mathfrak{z}_l{}^n \mathfrak{z}_l{}^*)\mathfrak{z}_l]\mathfrak{z}_l$

$= \delta[[(\mathfrak{z}_1{}^n \mathfrak{z}_1{}^*)\mathfrak{z}_1]\mathfrak{z}_1 \otimes \dots \otimes [(\mathfrak{z}_l{}^n \mathfrak{z}_l{}^*)\mathfrak{z}_l]\mathfrak{z}_l]$

$= \delta[[(\mathfrak{z}_1{}^n \mathfrak{z}_1{}^*)\mathfrak{z}_1 \otimes \dots \otimes (\mathfrak{z}_l{}^n \mathfrak{z}_l{}^*)\mathfrak{z}_l](\mathfrak{z}_1 \otimes \dots \otimes \mathfrak{z}_l)]$

$= \delta[((\mathfrak{z}_1{}^n \otimes \dots \otimes \mathfrak{z}_l{}^n)(\mathfrak{z}_1{}^* \otimes \dots \otimes \mathfrak{z}_l{}^*))(\mathfrak{z}_1 \otimes \dots \otimes \mathfrak{z}_l)](\mathfrak{z}_1 \otimes \dots \otimes \mathfrak{z}_l)$

$= [((\mathfrak{z}_1 \otimes \dots \otimes \mathfrak{z}_l)^n (\mathfrak{z}_1 \otimes \dots \otimes \mathfrak{z}_l)^*)(\mathfrak{z}_1 \otimes \dots \otimes \mathfrak{z}_l)](\mathfrak{z}_1 \otimes \dots \otimes \mathfrak{z}_l)$

Then, $(\mathfrak{z}_1 \otimes \dots \otimes \mathfrak{z}_l)$ is δ-qt-operator with order n.

Conclusion

In this work, we introduced the definition of a quasi-triple operator for a complex Hilbert space and generalized this definition to quasi triple operator of the nth order and δ-quasi triple operator with order n ($n \geq 1$), and gave some examples and properties about it as well as proving some facts about these operators.

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Conflict of interest

None.

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