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The Zagreb Coindices to the Zero Divisors Graph of Principal Ideal Local Rings

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Abstract

In this paper, we study the graph structure of the zero divisor graph $\Gamma(R)$, when R is a local principal ideal ring (P.I.R.). Special attention is given to the case when the nilpotency index t is an even or odd positive integer, and the graph structure is clearly given in terms of the properties of the sets X_i. Formulas are given for computing the Zagreb coindices of the graph $\Gamma(R)$. These results build an understanding of how the properties of algebraic ring are related to the structural graph properties of the respective graphs.

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1. Introduction

This paper discusses a commutative ring R with an identity element $1 \neq 0$, when R local principal ideal ring (P.I.R) with nilpotency equals any positive integer number. and $Z(R)$ denotes the collection of all zero divisors of the ring. In 1988, Beck (1) used graph theory to describe zero divisors. He created a graph in which any two elements s and r satisfying the relation $s.r = 0$ are represented as connected nodes (vertices), together with the zero element. Anderson and Livingston then extended this idea in 1999 (2) by eliminating the zero element. This reduced the graphs made better to use. Because of this change, a lot of researchers became interested in the topic. You can read more on this from references [3–6]. This study continues the line of previous work [7,8] that explored the zero-divisor graph $\Gamma(R)$ structure in similar contexts, follows old research and adds new things. This paper wants to find general formulas for the first and second Zagreb coindices of $\Gamma(R)$. R is a local principal ideal ring here, and the nilpotency index t is an even or odd positive integer. These coindices offer a different way of looking at Zagreb indices by examining structural properties based on pairs of vertices that are not adjacent. By differentiating between the vertex sets X_i and their relations to each other, we have formulas that allow us to more fully understand the algebraic structure of the ring given by its associated graph.

2. Preliminaries

We offer some basic definitions of ring theory and graph theory.

Definition 2.1 (9): Let R is a commutative ring. An ideal L in a ring R is called a maximal ideal if $L \neq R$ and for any ideal A of R such that $L \subset A \subseteq R$, then $A = R$. And the local ring contains only one maximal ideal and denoted by L

Definition 2.2 (10): An element u of a ring R such that, $ur = ru = 1$, where $r \in R$ is called a unit element. And the set of all $u \in R$ is denoted by $U(R)$.

Definition 2.3 (11): A local principal ideal ring P.I.R. is a ring where all ideals are principal and it has a unique maximal ideal L, where every non-invertible element belongs to the maximal ideal.

Definition 2.4 (12): A member a of a ring R is called nilpotent if there is some positive integer t with $a^t = 0$ where t is the least positive integer number, this property is called the nilpotency index of a.

Definition 2.5 (12): An ideal I in a ring R is called a nilpotent ideal if there is some positive integer m such that $I^m = \{0\}$, The least positive integer m such that the condition is called the nilpotency index of the ideal I.

Definition 2.6 (9): The order of a set refers to the number of elements in the set. It is also called the cardinality of the set. If a set S has n elements, its order is denoted as $| S | = n$.

Definition 2.7 (13): A graph G is formally defined as an ordered pair $G = (V, E)$, where $V(G) = \{v_1, v_2, v_3, \dots v_n\}$ is the set of vertices, and $E(G) = \{e_1, e_2, e_3, \dots e_m\}$, for all $n, m \in N$ is the collection of edges, each edge connecting a pair of graph vertices.

Definition 2.8 (13): The order of the graph G is denoted by $n(G)$ and is defined to be the number of vertices of the graph where $n(G) = |V(G)|$.

Definition 2.9 (14): The degree of a vertex v is denoted by $deg(v)$, is defined as the number of vertices adjacent to a given vertex v in a graph G.

Definition 2.10 (14): A graph G is said to be complete if every vertex of the graph is joined to every other vertex. That is, there is an edge between every pair of distinct vertices. A complete graph with n vertices is denoted by K_n.

Definition 2.11 (15,16): The first Zagreb coindices $\overline{M_1}$ is the sum of the degrees of all pairs of non-adjacent vertices in the graph. It is given by:

$$\overline{M_1} = \sum_{uv \notin E(G)} (deg(u) + deg(v))$$

Definition 2.12 (15,16): The second Zagreb coindex $\overline{M_2}$ is determined by summing the products of the degrees of all pairs of non-adjacent vertices of the graph:

$$\overline{M_2} = \sum_{uv \notin E(G)} (deg(u)\, deg(v)).$$

3. Main Results

In order to examine the structure of $\Gamma(R)$ of a local principal ideal ring $(\mathrm{R}, \mathrm{L}, \mathrm{t})$ where R is a ring, L is maximal ideal of R and t is a positive integer, we divided the vertex set into disjoint subsets $X_i \subseteq \Gamma(R)$. These subsets are formed according to the difference of the powers of the maximal ideal L. The subsets X_i, indexed by $i = 1, \dots, t-1$, possess different adjacency properties based on the value of i in comparison with t. So, the induced subgraphs of X_i belong to one of two types: nil-subgraphs or complete subgraphs

Lemma 3.1 (8): Let R be a local P.I.R with nilpotency t, where t any positive integer number. Then any two subsets X_i, X_j of $\Gamma(R)$ is adjacent if and only if $i + j \geq t$.

Remark 3.2: From above **(Lemma 3.1)**, one can see clearly that the properties of adjacency in the sets are as follows. The set X_1 adjacent to only X_{t-1}, and X_2 adjacent to X_{t-1} and X_{t-2} , to $X_{\frac{t}{2}-1}$ is adjacent to all sets X_i, when $i = \frac{t}{2} + 1, \dots, t-1$, for all t even or odd

Theorem 3.3: In local P.I.R with nilpotency t even positive integer number, the first Zagreb coindex is given by:

$$\overline{M_1} = \sum_{i=1}^{\left|X_{\frac{t}{2}-1}\right|} \left((|X_i|)(|X_i| - 1) \left(\sum_{j=t-i}^{t-1} |X_j| \right) \right)$$
$$+ \sum_{i=1}^{\left|X_{\frac{t}{2}-2}\right|} |X_i| \left(\sum_{k=i+1}^{\frac{t}{2}-1} \sum_{h=1}^{|X_k|} \left[\sum_{j=t-i}^{t-1} |X_j| + deg(u_h) \right] \right)$$
$$+ \sum_{i=1}^{\left|X_{\frac{t}{2}-1}\right|} |X_i| \left(\sum_{k=i+1}^{\frac{t}{2}-1} \sum_{h=1}^{|X_k|} \left(\sum_{j=t-i}^{t-1} |X_j| + deg\,(u_h^*) \right) \right)$$

Proof: From the above **Remark 3.2**, we have every vertex from X_1 to $X_{\frac{t}{2}-1}$ are not adjacent between them. Also, all vertices in X_{t-1} are excluded because all these vertices are adjacent to all other vertices of the graph $\Gamma(R)$. Therefore, to find the first Zagreb coindex, the sum of numbers of pair different vertices without repeating pairs, therefore we divided the proof to three parts.

Part 1. Used the combination law for choosing two vertices within each set separately $(i = j)$ to $X_1, \dots, X_{\frac{t}{2}-1}$,where all vertices on set have the same degree.

Since,

$$deg\ (v)_{v\in X_i} = \sum_{j=t-i}^{t-1} |X_j|\,, i = 1, \dots, \frac{t}{2} - 1$$

Apply that to the first Zagreb coindex low, there is:

$$\overline{\boldsymbol{M_1}}^* = C_2^{|X_1|}\left(2\,deg(v)_{v\in X_1}\right) + C_2^{|X_2|}\left(2\,deg(v)_{v\in X_2}\right) + \cdots + C_2^{\left|X_{\frac{t}{2}-1}\right|}\left(2\,deg(v)_{v\in X_{\frac{t}{2}-1}}\right),$$

$$= 2\left(\frac{(|X_1|)(|X_1|-1)}{2}\right)\sum_{j=t-1}^{t-1}|X_j| + \cdots + 2\left(\frac{\left(\left|X_{\frac{t}{2}-1}\right|\right)\left(X_{\frac{t}{2}-1}-1\right)}{2}\right)\sum_{j=t-\frac{t}{2}-1}^{t-1}|X_j|,$$

$$= (|X_1|)(|X_1|-1)\sum_{j=t-1}^{t-1}|X_j| + \cdots + \left(\left|X_{\frac{t}{2}-1}\right|\right)\left(X_{\frac{t}{2}-1}-1\right)\sum_{j=t-\frac{t}{2}-1}^{t-1}|X_j|$$

$$= \sum_{i=1}^{\frac{t}{2}-1}\left((|X_i|)(|X_i|-1)\left(\sum_{j=t-i}^{t-1}|X_j|\right)\right).$$

Part 2. We will analyze the cases of non-adjacency between the vertices of distinct sets $(i \neq j)$ form X_1 to $X_{\frac{t}{2}-1}$, where choosing two distinct vertices on set has a deferent degree.

When $v \in X_i$ and $u \in X_h$

$$A_1 = |X_1|\left(\sum_{h=1}^{|X_2|}\left(deg(v)_{v\in X_1} + deg(u_h)\right) + \cdots + \sum_{h=1}^{\left|X_{\frac{t}{2}-1}\right|}\left(deg(v)_{v\in X_1} + deg(u_h)\right)\right)$$

$$= |X_1|\left(\sum_{k=2}^{\frac{t}{2}-1}\sum_{h=1}^{|X_k|}\left(deg(v)_{v\in X_1} + deg\ (u_h)\right)\right)$$

$$\boldsymbol{A_2} = |X_2|\left(\sum_{h=1}^{|X_3|}\left(deg(v)_{v\in X_2} + deg(u_h)\right) + \cdots + \sum_{h=1}^{\left|X_{\frac{t}{2}-1}\right|}\left(deg(v)_{v\in X_2} + deg(u_h)\right)\right)$$

$$= |X_2|\left(\sum_{k=3}^{\frac{t}{2}-1}\sum_{h=1}^{|X_k|}\left(deg(v)_{v\in X_2} + deg\ (u_h)\right)\right)$$

$$\boldsymbol{A_3} = |X_3|\left(\sum_{h=1}^{|X_4|}\left(deg(v)_{v\in X_3} + deg(u_h)\right) + \cdots + \sum_{h=1}^{\left|X_{\frac{t}{2}-1}\right|}\left(deg(v)_{v\in X_3} + deg(u_h)\right)\right)$$

$$= |X_3|\left(\sum_{k=4}^{\frac{t}{2}-1}\sum_{h=1}^{|X_k|}\left(deg(v)_{v\in X_3} + deg\ (u_h)\right)\right)$$

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$$\boldsymbol{A_{\frac{t}{2}-2}} = \left|X_{\frac{t}{2}-2}\right|\sum_{h=1}^{|X_{t-1}|}\left(deg(v)_{v\in X_{\frac{t}{2}-2}} + deg(u_h)\right)$$

$$= \left|X_{\frac{t}{2}-2}\right|\sum_{k=\frac{t}{2}-1}^{\frac{t}{2}-1}\sum_{h=1}^{|X_k|}\left(deg(v)_{v\in X_{\frac{t}{2}-2}} + deg\ (u_h)\right)$$

Therefore, $\overline{\boldsymbol{M_1}}^{**} = A_1 + \cdots + A_{\frac{t}{2}-2}$

$$\overline{\boldsymbol{M_1}}^{**} = \sum_{i=1}^{\left|X_{\frac{t}{2}-2}\right|} |X_i| \left(\sum_{k=i+1}^{\frac{t}{2}-1} \sum_{h=1}^{|X_k|} \left(deg(v)_{v\in X_i} + deg\,(u_h) \right) \right)$$

Since,

$$deg\,(v)_{v\in X_i} = \sum_{j=t-i}^{t-1} \left|X_j\right|, i = 1, \ldots, \frac{t}{2} - 1$$

$$\overline{\boldsymbol{M_1}}^{**} = \sum_{i=1}^{\left|X_{\frac{t}{2}-2}\right|} |X_i| \left(\sum_{k=i+1}^{\frac{t}{2}-1} \sum_{h=1}^{|X_k|} \left(\sum_{j=t-i}^{t-1} \left|X_j\right| + deg(u_h) \right) \right)$$

And used the degree of the vertices of $deg\,(u_h)$ law after the distribution of all summation, where

$$de\,g(u_h) = \sum_{j=t-i}^{t-1} \left|X_j\right|, i = 1, \ldots, \frac{t}{2} - 1$$

Part 3. Now, we consider the cases of non-adjacency between the elements of the sets X_i, where

$$i = 1, \ldots, \frac{t}{2} - 1$$

And the elements of the sets X_j, where

$$j = \frac{t}{2} + \frac{t}{2} + 1, \ldots, t - 2$$

Now,

$$\boldsymbol{B_1} = |X_1| \left(\sum_{h=1}^{\left|X_{\frac{t}{2}}\right|} \left(deg(v)_{v\in X_1} + deg\,(u_h^*) \right) + \cdots + \sum_{h=1}^{|X_{t-2}|} \left(deg(v)_{v\in X_1} + deg\,(u_h^*) \right) \right)$$

$$= |X_1| \left(\sum_{k=\frac{t}{2}}^{t-2} \sum_{h=1}^{|X_k|} \left(deg(v)_{v\in X_1} + deg\,(u_h^*) \right) \right)$$

$$\boldsymbol{B_2} = |X_2| \left(\sum_{h=1}^{\left|X_{\frac{t}{2}}\right|} \left(deg(v)_{v\in X_2} + deg\,(u_h^*) \right) + \cdots + \sum_{h=1}^{|X_{t-3}|} \left(deg(v)_{v\in X_2} + deg\,(u_h^*) \right) \right)$$

$$= |X_2| \left(\sum_{k=\frac{t}{2}}^{t-3} \sum_{h=1}^{|X_k|} \left[deg(v)_{v\in X_2} + deg\,(u_h^*) \right] \right)$$

$$\boldsymbol{B_3} = |X_3| \left(\sum_{h=1}^{\left|X_{\frac{t}{2}}\right|} \left(deg(v)_{v\in X_3} + deg\,(u_h^*) \right) + \cdots + \sum_{h=1}^{|X_{t-4}|} \left(deg(v)_{v\in X_3} + deg\,(u_h^*) \right) \right)$$

$$= |X_3| \left(\sum_{k=\frac{t}{2}}^{t-4} \sum_{h=1}^{|X_k|} \left(deg(v)_{v\in X_3} + deg\,(u_h^*) \right) \right)$$

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$$\boldsymbol{B_{\frac{t}{2}-1}} = \left|X_{\frac{t}{2}-1}\right| \left(\sum_{h=1}^{\left|X_{\frac{t}{2}}\right|} \left(deg(v)_{v\in X_{\frac{t}{2}-1}} + deg\,(u_h^*) \right) \right)$$

$$= \left|X_{\frac{t}{2}-1}\right| \left(\sum_{k=\frac{t}{2}-1}^{\frac{t}{2}-1} \sum_{h=1}^{|X_k|} \left(deg(v)_{v\in X_{\frac{t}{2}-1}} + deg\,(u_h^*) \right) \right)$$

Therefore,

$$\overline{\boldsymbol{M_1}}^{***} = B_1 + \cdots + B_{\frac{t}{2}-1}$$

$$= \sum_{i=1}^{\left|X_{\frac{t}{2}-1}\right|} |X_i| \left(\sum_{k=i+1}^{\frac{t}{2}-1} \sum_{h=1}^{|X_k|} \left(deg(v)_{v \in X_i} + deg\,(u_h^*) \right) \right)$$

$$= \sum_{i=1}^{\left|X_{\frac{t}{2}-1}\right|} |X_i| \left(\sum_{k=i+1}^{\frac{t}{2}-1} \sum_{h=1}^{|X_k|} \left(\sum_{j=t-i}^{t-1} |X_j| + deg\,(u_h^*) \right) \right)$$

And used the degree of the vertices of $deg(u_h)^*$ law after the distribution of all the summation

$$deg(u_h^*) = \sum_{j=t-i}^{t-1} |X_j| - 1, i = \frac{t}{2}, \dots, t-1$$

Hence,

$$\overline{M_1} = \overline{M_1}^{*} + \overline{M_1}^{**} + \overline{M_1}^{***}$$

$$\overline{\boldsymbol{M_1}} = \sum_{i=1}^{\left|X_{\frac{t}{2}-1}\right|} \left((|X_i|)(|X_i| - 1) \left(\sum_{j=t-i}^{t-1} |X_j| \right) \right)$$

$$+ \sum_{i=1}^{\left|X_{\frac{t}{2}-2}\right|} |X_i| \left(\sum_{k=i+1}^{\frac{t}{2}-1} \sum_{h=1}^{|X_k|} \left(\sum_{j=t-i}^{t-1} |X_j| + deg(u_h) \right) \right)$$

$$+ \sum_{i=1}^{\left|X_{\frac{t}{2}-1}\right|} |X_i| \left(\sum_{k=i+1}^{\frac{t}{2}-1} \sum_{h=1}^{|X_k|} \left(\sum_{j=t-i}^{t-1} |X_j| + deg\,(u_h^*) \right) \right)$$

Similarly, the first Zagreb coindex where t an odd positive integer number can be derived taking into account the non adjacency properties between the sets, therefore the adjacency formula is as given:

$$\overline{\boldsymbol{M_1}} = \sum_{i=1}^{\frac{t-1}{2}} \left((|X_i|)(|X_i| - 1) \left(\sum_{j=t-i}^{t-1} |X_j| \right) \right)$$

$$+ \sum_{i=1}^{\left|X_{\frac{t-1}{2}-1}\right|} |X_i| \left(\sum_{k=i+1}^{\frac{t-1}{2}} \sum_{h=1}^{|X_k|} \left(\sum_{j=t-i}^{t-1} |X_j| + deg(u_h) \right) \right)$$

$$+ \sum_{i=1}^{\left|X_{\frac{t-1}{2}}\right|} |X_i| \left(\sum_{k=i+1}^{\frac{t-1}{2}} \sum_{h=1}^{|X_k|} \left(\sum_{j=t-i}^{t-1} |X_j| + deg\,(u_h^*) \right) \right).$$

Theorem 3.4: In local P.I.R with nilpotency (t) even positive integer number, the second Zagreb coindex is given by:

$$\overline{\boldsymbol{M_2}} = \sum^{\frac{t}{2}-1}_{i=1} \left(\frac{(|X_i|)(|X_i| - 1)}{2} \left(\sum_{j=t-i}^{t-1} |X_j| \right)^2 \right)$$

$$+ \sum_{i=1}^{\left|X_{\frac{t}{2}-2}\right|} |X_i| \left(\sum_{k=i+1}^{\frac{t}{2}-1} \sum_{h=1}^{|X_k|} \left(\sum_{j=t-i}^{t-1} |X_j| . deg(u_h) \right) \right)$$

$$+ \sum_{i=1}^{\left|X_{\frac{t}{2}-1}\right|} |X_i| \left(\sum_{k=i+1}^{\frac{t}{2}-1} \sum_{h=1}^{|X_k|} \left(\sum_{j=t-i}^{t-1} |X_j| . deg\,(u_h^*) \right) \right)$$

Proof: From above **Remark 3.2,** we have every vertex from X_1 to $X_{\frac{t}{2}-1}$ are not adjacent between them. Also, all vertices in X_{t-1} are excluded because all these vertices are adjacent to all another vertex of graph $\Gamma(R)$. Therefore, to find the second Zagreb coindex, the sum of product numbers of pair different vertices without repeating pairs, therefore we divided the proof to three parts.

Part 1. The union rule was applied in selecting pairs of vertices from within each provided set (i.e., for $i = j$), that is, from sets $X_1, \dots, X_{\frac{t}{2}-1}$, whose vertices possess the same degree. Since,

$$deg(v)_{v \in X_i} = \sum_{j=t-i}^{t-1} |X_j|, i = 1, \dots, \frac{t}{2} - 1$$

Apply that to the second Zagreb coindex low, there is :

$$\overline{\boldsymbol{M_2}}^{*} = C_2^{|X_1|} \left(deg(v)_{v \in X_1} \right)^2 + \cdots + C_2^{\left|X_{\frac{t}{2}-1}\right|} \left(deg(v)_{v \in X_{\frac{t}{2}-1}} \right)^2$$

$$= \frac{(|X_1|)(|X_1| - 1)}{2} \left(\sum_{j=t-1}^{t-1} |X_j| \right)^2 + \cdots$$

$$+ \frac{\left(\left|X_{\frac{t}{2}-1}\right|\right)\left(X_{\frac{t}{2}-1} - 1\right)}{2} \left(\sum_{j=t-\frac{t}{2}-1}^{t-1} |X_j| \right)^2$$

$$= \sum_{i=1}^{\frac{t}{2}-1}\left(\frac{(|X_i|)(|X_i|-1)}{2}\left(\sum_{j=t-i}^{t-1}|X_j|\right)^2\right)$$

Similarly, applying the **second** and **third** parts of (**Theorem 3.3**) repeatedly, we have

$$\overline{\boldsymbol{M_2}}^{**} = \sum_{i=1}^{\left|X_{\frac{t}{2}-2}\right|}|X_i|\left(\sum_{k=i+1}^{\frac{t}{2}-1}\sum_{h=1}^{|X_k|}\left(\sum_{j=t-i}^{t-1}|X_j|.deg\,(u_h^*)\right)\right)$$

And,

$$\overline{\boldsymbol{M_2}}^{***} = \sum_{i=1}^{\left|X_{\frac{t}{2}-1}\right|}|X_i|\left(\sum_{k=i+1}^{\frac{t}{2}-1}\sum_{h=1}^{|X_k|}\left(\sum_{j=t-i}^{t-1}|X_j|.deg\,(u_h^*)\right)\right)$$

Hence,

$$\overline{\boldsymbol{M_2}} = \overline{M_2}^{*} + \overline{M_2}^{**} + \overline{M_2}^{***}$$

So,

$$\overline{\boldsymbol{M_2}} = \sum_{i=1}^{\frac{t}{2}-1}\left(\frac{(|X_i|)(|X_i|-1)}{2}\left(\sum_{j=t-i}^{t-1}|X_j|\right)^2\right) + \sum_{i=1}^{\left|X_{\frac{t}{2}-2}\right|}|X_i|\left(\sum_{k=i+1}^{\frac{t}{2}-1}\sum_{h=1}^{|X_k|}\left(\sum_{j=t-i}^{t-1}|X_j|.deg(u_h)\right)\right) + \sum_{i=1}^{\left|X_{\frac{t}{2}-1}\right|}|X_i|\left(\sum_{k=i+1}^{\frac{t}{2}-1}\sum_{h=1}^{|X_k|}\left(\sum_{j=t-i}^{t-1}|X_j|.deg\,(u_h^*)\right)\right)$$

Similarly, the second Zagreb coindex, where t an odd positive integer number can be derived taking into account the non-adjacency properties between the sets, where the adjacency formula is as given:

$$\overline{\boldsymbol{M_2}} = \sum_{i=1}^{\frac{t}{2}-1}\left(\frac{(|X_i|)(|X_i|-1)}{2}\left(\sum_{j=t-i}^{t-1}|X_j|\right)^2\right) + \sum_{i=1}^{\left|X_{\frac{t-1}{2}-1}\right|}|X_i|\left(\sum_{k=i+1}^{\frac{t-1}{2}}\sum_{h=1}^{|X_k|}\left(\sum_{j=t-i}^{t-1}|X_j|.deg(u_h)\right)\right) + \sum_{i=1}^{\left|X_{\frac{t-1}{2}}\right|}|X_i|\left(\sum_{k=i+1}^{\frac{t-1}{2}}\sum_{h=1}^{|X_k|}\left(\sum_{j=t-i}^{t-1}|X_j|.deg\,(u_h^*)\right)\right).$$

This way of classifying the sets shows how non-zero zero divisors of R are arranged in layers based on the powers of L. This also helps us understand exactly how the vertices are connected and, in general, how the graph $\Gamma(R)$ is organized.

Conclusion

This paper provides general formulas for the Zagreb coindices of $\Gamma(R)$ in local principal ideal rings. By vertex subset analysis, it shows how algebraic ring structures determine the non-adjacency of the graph and overall structure.

Conflict of interest

None.

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