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A Novel Approach to Calculating the Riemann Mapping Function Using Real-Valued Kernels

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Abstract

This paper discusses the general boundary integral equations, Riemann mapping function for bounded simply connected regions whose boundary curves are smooth and analytic, that correspond to a real-valued kernel, and a new boundary integral formula is introduced. Based on the multiplication of the Kerzman-Stien Trummer integral formula (briefly, KSTiF) by the penalty function, such that the complex-valued kernel transforms into, we derive a new formula and prove its uniqueness. Numerical results using the Nyström method with the trapezoidal rule yield approximations of high accuracy when compared with exact solutions for test regions.

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1. Introduction

Conformal maps are essential for various contemporary technological issues. Utilizing conformal mapping functions, we can transform a complicated region into a fundamental region. This process has been applied in fluid flow, neuroscience, electrostatics, heat conduction, mechanics, and aerodynamics. For instance, neuroscience recently [1] used conformal mapping to investigate how large networks of spiking neurons self-organize in time to process and encode information in the brain. For more details [2-5]. The Riemann mapping theorem asserts the existence and uniqueness of a conformal mapping of a simply connected region in the complex plane onto the unit disk. Since the analytic approach is limited except for some special regions, the numerical approach is required in numerous applied issues, e.g., [1, 6, 7]. One of the preferred numerical methods in the numerical conformal mapping is the boundary integral formula method [6-11]. The integral formula method is a flexible technique to solve the Riemann–Hilbert issue in simply connected regions in the complex plane, see e.g., [7, 12]. For the uniqueness of interior Riemann-Hilbert issues, see [9, 10]. According to [11], one boundary integral formula has a real-valued kernel derived from the Bergman kernel's integral formula, which was developed by [12]. [13] sought a general boundary relationship that embodies both boundary, interior, and exterior relationships as special cases. A new way of looking at previous studies, as well as the advancement of knowledge, is the most important aspect of this dissertation. However, the Riemann mapping function is an effective way to transform any simply connected region to the unit disc. Since an analytic approach is sometimes invalid for complicated regions, numerical approximation methods are the classical alternative solution to solve many applications in science and engineering.

However, based on the KSTiF [6], there is no new derivation yet. Moreover, express "The computing time is multiplied by a factor of 4 concerning the usual real integral formula in conformal mapping". Therefore, the notion behind deriving a new boundary integral formula with a real-valued kernel from the KSTiF is to reduce time consumption by transforming the complex kernel into a real one. Thus, analytically, this paper contributes to deriving a new boundary integral formula with a real-valued kernel based on the KSTiF. In addition, the resolvability is lectured on with a proven theorem. The remaining vectors of this paper are organized as follows: Section 2 describes the KSTiF [6]. The new proposed method is introduced in Section 3. The numerical implementations are presented in Section 4. Results and comparison will be presented in Section 5. Finally, in Section 6, we summarize the main contributions and provide some future work.

2. The KSTiF Method

Let Ω be a bounded simply connected region on the complex plane ℂ whose boundary Γ is assumed to be an analytic Jordan curve. The section will discuss some functions related to the SzegÖ kernel δ (z, a). Suppose boundary Γ counter-clockwise parameterization z(t), $0 \le t \le \beta$, with $z'=dz/dt \neq 0$, ∀t. The unit tangent to boundary Γ at the point z(t) will be T(z) =z′/|z′|. In terms of solving the conformal mapping issues, it's sufficient to compute the boundary values of the SzegÖ kernel. SzegÖ kernel can be computed as a solution of the second kind Fredholm integral formula, as stated in the following theorem.

Theorem 1*:* [6]The SzegÖ kernel δ (z, a) is the unique continuous solution of the KSTiF.

$$\delta(z, a) + \int_{\Gamma} A(z,\omega)\ \delta(\omega,a)\ |d\omega| = \overline{H(a,z)},\ z \in \Gamma,\ a \in \Omega, \tag{1}$$

where

$$A(z,\omega) = \begin{cases} \overline{H(\omega,z)} - H(z,\omega), & if\ \omega \neq z,\ a \in \Omega \\ 0, & if\ \omega = z,\ a \in \Omega \end{cases} \tag{2}$$

and

$$H(\omega,z) = \frac{1}{2\pi i}\frac{T(z)}{z-\omega},\ \omega \in \overline{\Omega},\ z \in \Gamma. \tag{3}$$

Let the boundary Γ be an analytic Jordan curve and smooth. The unit tangent to Γ obtained by T(z) =z′/|z′| such that $z'=dz/dt \neq 0$. Let a ∈ Ω be a fixed point, while R be the RM normalized at a, that is

$$R(a) = 0,\ R'(a) > 0. \tag{4}$$

There are some required formulas for achieving this section:

$$R(z) = \frac{1}{i}T(z)\frac{\acute{R}(z)}{|\acute{R}(z)|},\quad z \in \Gamma, \tag{5}$$

$$\acute{R}(z) = \frac{2\pi}{S(a,a)}\delta(z,a)^2,\quad z \in \overline{\Omega}, \tag{6}$$

$$R(z) = \frac{1}{i}T(z)\frac{\delta(z,a)^2}{|\delta(z,a)^2|},\quad z \in \Gamma. \tag{7}$$

For further aspects about the KSTiF method, see [4, 14].

3. The new KSTiF Method

Based on the limitations of the previous method, which requires solving the complex system, which can be rewritten as a $2n \times 2n$ real system. The authors of [6] state: "The computing time is multiplied by a factor of 4 concerning the usual real integral formula in Conformal napping". Consequently, this section shows the selected penalty function, which reduces the complex real value into the real-value kernel. Thus, instead of solving the $2n \times 2n$ real system, the new KSTiF solves the $n \times n$ real system and requires almost a quarter of the time of the KSTiF method in computation. By multiplying the equation (1) by $T(z)^{1/2}$ yields.

$$T(z)^{\frac{1}{2}}\ \delta(z,\ a) - \int_{\Gamma} A_1(z,\omega)T(z)^{\frac{1}{2}}\delta(\omega,a)\ |d\omega| = T(z)^{\frac{1}{2}}\ \overline{H(a,z)},\ z \in \Gamma,\quad a \in \Omega, \tag{8}$$

$$A_1(z,\omega) = \begin{cases} \frac{1}{\pi}\left[\frac{T(z)^{\frac{1}{2}}\,T(\omega)^{\frac{1}{2}}}{\omega - z}\right], & if\ \omega \neq z \in \Gamma, \\ 0, & if\ \omega = z \in \Gamma. \end{cases} \tag{9}$$

Theorem 2. The function $\hat{\delta}(z, a) = T(z)^{\frac{1}{2}}\delta(z.a)$ is the unique continuous solution of the integral formula.

$$\hat{\delta}(z,a) - \int_{\Gamma} A_1(z,\omega)\hat{\delta}(\omega,a)\ |d\omega| = T(z)^{\frac{1}{2}}\overline{H(a,z)}, z \in \Gamma. \tag{10}$$

Proof:

Since $A_1(z, \omega)$ is a real kernel, it's enough to prove that it is a skew-symmetric and continuous function on Γ×Γ, we have

$$A_1(\omega,z) = \frac{1}{\pi}\mathrm{Im}\left[\frac{T(\omega)^{\frac{1}{2}}\,T(z)^{\frac{1}{2}}}{z-\omega}\right] = -\left[\frac{T(z)^{\frac{1}{2}}\,T(\omega)^{\frac{1}{2}}}{\omega - z}\right] = -A_1(z,\omega).$$

Then $A_1(z, \omega)$ is skew-symmetric on Γ × Γ. From the definition of continuity, we
Have

$$\lim_{\omega \to z} A_1(z,\omega) = A_1(z,z) = 0.$$

Thus, $A_1(z, \omega)$ is a continuous function on Γ × Γ. Since the kernel $A_1(z, \omega)$ is continuous and skew-symmetric on Γ × Γ, then according to [6], the eigenvalues are purely imaginary, and the associated integral formula (10) has a unique solution. Using equation (7), the boundary values of the Riemann map R are evaluated in terms of $\hat{\delta}$ (z, a) by

$$R(z) = \frac{1}{i}\frac{\hat{\delta}(z,a)^2}{|\hat{\delta}(z,a)^2|},\quad z \in \Gamma. \tag{11}$$

4. The Numerical Implementation of the New KSTiF (10)

In this section, we concentrate on the numerical implementations of the integral formula (10). Using a parametric representation of *z(t)* of Γ, $z = z(t),\ 0 \le t \le \beta$. With $\omega = z(s)$, then the integral formula (10) yields

$$\eta(t) - \int_0^{\beta} v(t,s)\eta(s)ds = g(t) = g(t), \tag{12}$$

where for $0 \le t \le 2\pi$,

$\eta(t) = |\acute{z}(t)|^{\frac{1}{2}}\ \hat{\delta}$ (z(t),0),

$v(t,s) = \sqrt{|\acute{z}(s)|}\sqrt{\acute{z}(t)}A_1(t,s) =$

$$\begin{cases} \frac{1}{\pi}\mathrm{Im}\left|\frac{\sqrt{|\acute{z}(s)|}\sqrt{\acute{z}(t)}}{z(s)-z(t)}\right|, & if\ t \neq s \in \Gamma, \\ 0, & if\ t = s \in \Gamma. \end{cases}$$

$$g(s) = \frac{-1}{2\pi\mathrm{i}}\frac{\sqrt{\overline{z(t)}}}{z(t)}.$$

Selecting η equidistant connection points $t_i=(i-1)\frac{\beta}{n}$, i=1,2,3,...n, we apply Nyström's method with the trapezoidal rule [15] to discretize equation (12), which yields

$$\eta(t_i) - \frac{\beta}{n}\sum_{j=1}^{n} v(t_i,t_j)\eta(t_j) = g(t_i), \qquad (13)$$

The trapezoidal rule is the most accurate method for integrating periodic functions [16, 17]. Let **B** be a matrix such that $B_{i,j} = \frac{\beta}{n}v(t_i,t_j)$ and let $x_i = \eta(t_i)$, $y_i = g(t_i)$.

Then equation (13) can be reduced to an n × n real system

$$(\mathbf{I} - \mathbf{B})x = y. \qquad (14)$$

Upon computation of the solution $xi = \eta(t_i)$, discretization of equation (13) yields an excellent "natural" interpolation formula:

$$\eta(t) = g(t) + \frac{\beta}{n}\sum_{j=1}^{n} v(t_i,t_j)\eta(t_j). \qquad (15)$$

Assume $\theta(t)$ represents the boundary of the correspondence formula for a parametric approximation $z(t)$, where t ranges from 0 to β of Γ. Therefore

$$\mathrm{R}(s(t)) = e^{-\mathrm{i}\theta}. \qquad (16)$$

Using (11), the formula, we can calculate the boundary correspondence function

$$\theta(t) = \arg(-\mathrm{i}\emptyset z^2). \qquad (17)$$

5. Results and Discussion

The numerical scheme discussed above applies to numerous test areas using normalization. Using Mathematica 15's "Linear Solve" tool, the system of equations (14) is solved. We list the sup-norm error $\|\phi(t)-\phi_n(t)\|_\infty$, where $\theta(t)$ are the exact boundary values and $\theta_n(t)$ are the approximations obtained using the interpolation formula (17).

High-accuracy results for both methods compared with exact solutions are presented in this section for some test regions. Moreover, we demonstrate the differences between the KSTiF and the new KSTiF in terms of accuracy, efficiency, and computational speed. Two typical examples have been selected as examples to compare the two methods, which have exact solutions, namely the Ellipse and Epitrochoid ("Apple"). Tables 1 and 2 demonstrate the error norms for both methods compared with exact solutions, as well as demonstrate the comparison results.

Example 1. "Ellipse" ($0 \leq \epsilon < 1$, axis ratio $=\frac{1+\epsilon}{1-\epsilon}$)

$$z(t) = e^{\mathrm{i}t} - \alpha e^{-\mathrm{i}t}$$

$$\theta(t) = t + 2\sum_{j=1}^{n}\frac{-1^k}{k}\frac{\epsilon^k}{1+\epsilon^{2k}}\sin 2kt.$$

Table 1. Comparison of the present method with the KST method for the ellipse: infinity-norm error $\|\emptyset(t) - \emptyset_n(t)\|_\infty$, on boundary Γ (Example 1).

axis ratio α						
n	**New KSTiF Method**			**KSTiF method**		
n	2	3	5	2	3	5
8	1.3(-02)	1.5(-01)	1.92	1.6(-02)	2.5(-01)	3.6
16	1.8(-04)	5.4(-05)	4.2(-02)	1.9(-04)	1.3(-02)	1.7
32	2.9(-08)	8.2(-10)	6.3(-05)	3.5(-07)	5.2(-05)	4.0(-02)
64	1.7(-11)	3.2(-11)	3.4(-06)	6.4(-09)	1.1(-07)	2.7(-06)

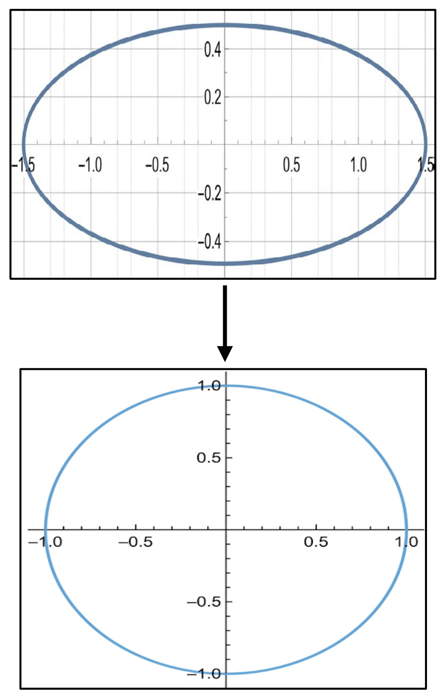

Figure 1. Conformal map for a bounded Ellipse simply domain with (axis ratio=3) to the boundary of a unit disc. Example 1.

Example 2. Epitrochoid "Apple" (0 ≤ α < 1)

$z(t) = e^{it} + 0.5\ \alpha\ e^{2it}$, $0 \leq t \leq 2\pi$,

θ(t) = t.

Table2. Comparison of the present method with the KST method for the 'Apple' epitrochoid: infinity-norm error $\| \emptyset(t) - \emptyset_n(t) \|_\infty$ on the boundary Γ (Example 2).

	axis ratio α					
	New KSTiF Method			KSTiF method		
n	0.3	0.6	0.9	0. 3	0.6	0.9
8	1.2(-02)	8.1 (-01)	9.8(-02)	1.4(-04)	1.0(-02)	2.1(-01)
16	1.1(-06)	8.7(-04)	6.4(-02)	1.7(-06)	8.4(-04)	7.4(-02)
32	1.9 (-11)	3.8(-06)	1.5(-02)	.2(-06)	4.1(-06)	1.9(-02)
64	1.3(-15)	3.6(-14)	4.4(-06)	4.8(-08)	4.8(-09)	3.2(-02)

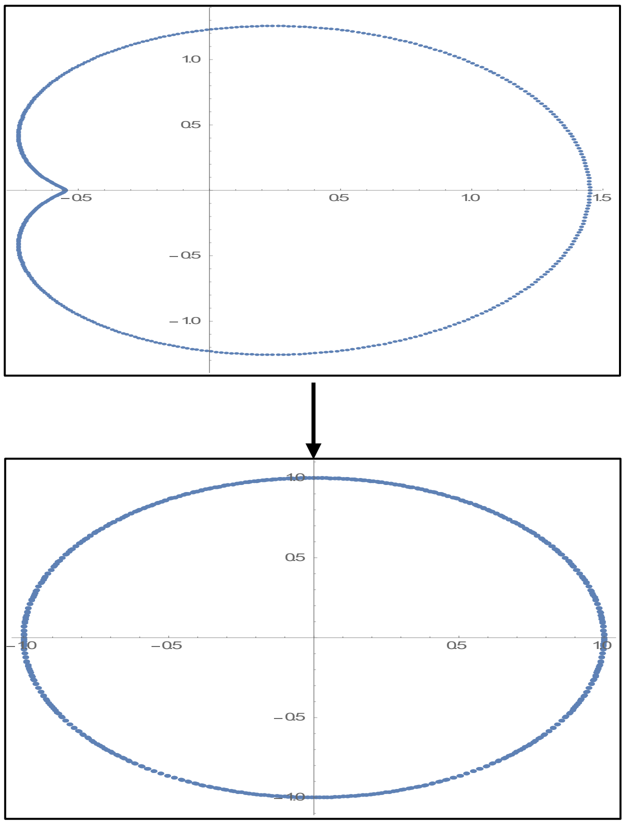

Figure 2. Conformal transformation of the 'Apple' epitrochoid boundary (axis ratio 0.9) to the unit circle.

Table 1 demonstrates that, for the elliptic case, more points are required to obtain good accuracy when the axis ratio increases. The reason is that the ellipse becomes narrower when the axis ratio increases.

The same reasoning moreover applies to the other 2 regions. Moreover, both **Tables 1 and 2** demonstrate that the KSTiF and new KSTiF methods are equivalent in terms of accuracy; however, in some cases, the new KSTiF method yields better results, which proves the robustness of the new KSTiF method. The aforementioned results illustrate that the new method is effective and robust, and additionally eliminates the shortcomings of the complexity and time-consuming nature of the KSTiF. Furthermore, the biggest advantage of the new KSTiF method is the real-value kernel, which reduces the computational efforts and then increases the computational speed of the numerical results by approximately four times [6]. Recent results offered by [8] describe an alternative method based on an integral equation with the classical Neumann kernel N(z, ω) real-value kernel, demonstrating that their method yields comparable accuracy to the method based on the SzegÖ kernel provided in [6], and requires less computational effort. This leads us to the question of whether our new method is equivalent to the recent method proposed by [8] in terms of accuracy and computational speed.

Therefore, since the new KSTiF method has the kernel $A_1(z, \omega)$, which is real and skew-Hermitian, we can express that it is a better method than the existing integral equation schemes mentioned in this paper, which yield high-performance results and overcome the previous methods' weaknesses.

Table 1 and **Table 2** shows that the present method proposed in this section seems to give better accuracy when the axis ratio α decreases.

However, in terms of computational speed, we do not need to compare with models that have a complex-valued kernel, as already addressed by [6]. Thus, we compare our new KSTiF method with the modified integral formula of the Bergman kernel $\hat{B}$ (z, α) [10]. One of our aims is to solve the new boundary integral equations and compare the numerical conformal mapping results with exact solutions for some selected regions. The integral equation (10) in Theorem 2 yields approximations of high accuracy when compared with precise solutions for four different test regions. The numerical results are comparable to the method based on the Szegö kernel.

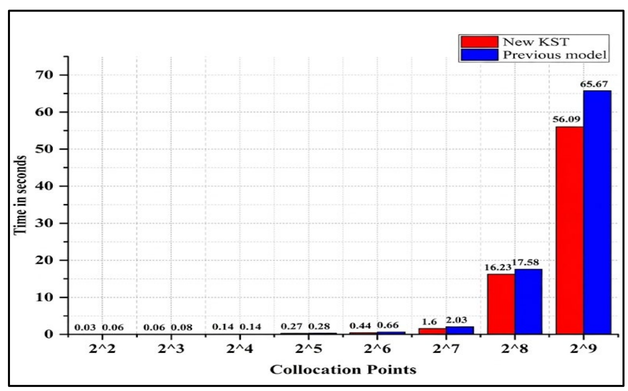

Figure 3. Time (secs) between the modified integral formula of the Bergman kernel $\hat{B}$(z, α) and the new KSTiF method.

From **Figure 3** it can be observed that our new KSTiF method is faster than the modified integral equation with Bergman kernel especially when n is large.
Moreover, the new KSTiF method has a kernel that is skew-symmetric, we do not need a full matrix to calculate, but rather to the fact that only the lower triangle enough since the points analogues.

Table 3. Disadvantages of the integral formula of $\hat{B}$ (z, α) versus the advantages of the new Kerzman-Stien Trummer scheme

Disadvantages of the integral formula of $\hat{B}$ (z, α) [10]	**Advantages of the New-Kerzman-Stien Trummer scheme**
1 Non-symmetric in general.	1 Skew-symmetric.
2 Require the second derivative.	2 First derivative only.
3 Require calculating all kernel points.	3 It's enough to calculate the lower triangle of the kernel matrix only.
4 The diagonal of the kernel matrix is not zero in general.	4 Diagonal elements of the kernel matrix are zeros.

Figure 3 demonstrates the execution time in the modified integral formula of the Bergman kernel $\hat{B}$ (z, α) [10] and our method. Other advantages are clearly explained in **Table 3.**

Conclusion

This paper presents a uniquely solvable numerical mathematical model to compute the conformal mapping of a simply connected regions from interior regions in the complex plane onto the unit disk. Numerical examples demonstrated that the new KSTiF method yields approximations with similar precision to the prior approach. Unlike the KSTiF model, the proposed method is characterized by possessing a real kernel and symmetry properties. In terms of efficiency, accuracy, computational speed, uniqueness, and kernel type, the new KSTiF method robustly satisfies all of these characteristics. The presented notion of transforming a complex real value into a real-value kernel can be extended to interior region of simply connected regions and may include multiple connected regions. Moreover, motivated by the results of [10], who developed a general boundary integral formula that embodies both boundary integral formulas [6, 10] as special cases, An interesting future part that can be settled is to transform its complex-valued kernel into a real-valued kernel such that our present method can be a special case. Additionally, proof in terms of the solvability of the general boundary integral formula given in [10] is still an open problem. The method arises from its application in mathematical neuroscience [18].

Conflict of interest

No conflict of interest

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